Notes on supply and demand.

Q: What's with the dummy factories and warehouses?

Consider the general problem with n factories and m warehouses. If the

$$\text{total supply} = \sum_{i=1}^{n} s_i \geq \sum_{j=1}^{m} d_j = \text{total demand}$$

then the following linear programming problem should make sense:

$$\begin{cases} \text{minimize} \quad \sum_i \sum_j c_{ij} x_{ij} \\ \text{subject to} \\ \qquad \sum_{j=1}^{m} x_{ij} \leq s_i \quad \text{for } i = 1 \ldots n \\ \qquad \sum_{i=1}^{n} x_{ij} \geq d_j \quad \text{for } j = 1 \ldots m \\ x_{ij} \geq 0 \quad \forall i, j \end{cases}$$

i.e. each factory ships no more than it has and each warehouse receives at least what it needs

How do we know there are any feasible solutions at all? Maybe the problem's infeasible ...

If you sum the supply constraints then

$$\sum_i \sum_j x_{ij} \le \sum_i s_i = \text{total supply.} \quad (*)$$

If you sum the demand constraints then

$$\text{total demand} = \sum_j d_j \le \sum_j \sum_i x_{ij} \quad (**)$$

i.e. $$\sum_j d_j \le \sum_j \sum_i x_{ij} \le \sum_i s_i$$

this suggests that if total demand $\le$ total supply then there's a hope for a feasible solution.

Case 1: total supply = total demand

i.e. $$\sum_j d_j = \sum_i s_i = S$$

In this case $x_{ij} = \dfrac{s_i d_j}{S}$ is a feasible solution.

since $$\sum_i x_{ij} = \sum_i \frac{s_i d_j}{S} = d_j \frac{\sum s_i}{S} = d_j \checkmark$$

$$\sum_j x_{ij} = \sum_j \frac{s_i d_j}{S} = s_i \frac{\sum_j d_j}{S} = s_i \checkmark$$

Also, as I proved in class, if

$$\sum_j d_j = \sum_i s_i$$

then the inequalities must all be equalities i.e. the linear programming problem is really

$$\begin{cases} \text{minimize} \quad \sum_i \sum_j c_{ij} x_{ij} \\ \text{subject to} \\ \qquad \sum_i x_{ij} = d_j \qquad j = 1 \dots m \\ \qquad \sum_j x_{ij} = s_i \qquad i = 1 \dots n \\ x_{ij} \geq 0 \quad \forall\, i,j \end{cases}$$

Case 2: total supply > total demand

We expect there should be a feasible solution

let $D = \sum_j d_j \qquad S = \sum_i s_i$.

By assumption, $D < S \quad \Rightarrow \quad \frac{D}{S} < 1$.

then $\quad x_{ij} = \frac{s_i d_j}{S}$ will be feasible since

$$\sum_i x_{ij} = d_j \frac{S}{S} = d_j \geq d_j \quad \checkmark$$

$$\sum_j x_{ij} = s_i \frac{D}{S} \leq s_i \quad \checkmark$$

So the system has feasible solutions.
I'll now argue something stronger. i.e.

Theorem: if $\{x_{ij}\}$ is an optimal solution of

$$\min \sum\sum c_{ij} x_{ij}$$

where
$$\sum_i x_{ij} \geq d_j$$
$$\sum_j x_{ij} \leq s_i$$
$$x_{ij} \geq 0$$

and $D = \sum_j d_j < \sum_i s_i = S$ then we have no slack in the demand equations.

Proof: Assume we have slack in one of the demand equations. i.e. $\sum_{i=1}^{n} x_{ij_0} > d_{j_0}$ for some j_0.

Choose $\varepsilon > 0$ so that $\sum_{i=1}^{n} x_{ij_0} - \varepsilon > d_{j_0}$. (You can do this if ε is small enough.) We use ε to introduce a new feasible solution $\{\tilde{x}_{ij}\}$ where

$$\tilde{x}_{1j_0} = x_{1j_0} - \varepsilon$$
$$\tilde{x}_{ij} = x_{ij} \quad \text{if } (i,j) \neq (1, j_0)$$

Check that $\{\tilde{x}_{ij}\}$ is still feasible... Since the only change was in $\tilde{x}_{1j_0}$, we just have to check the first supply inequality and the j_0th demand inequality:

$$\sum_{j=1}^{m} \tilde{x}_{1j} = \left(\sum_{j=1}^{m} x_{1j}\right) - \varepsilon \le s_1 - \varepsilon \quad \text{(since } \{x_{ij}\} \text{ is feasible)}$$

$$\le s_1 \checkmark \quad \text{(since } \varepsilon > 0\text{)}$$

$$\sum_{i=1}^{n} \tilde{x}_{ij_0} = \left(\sum_{i=1}^{n} x_{ij_0}\right) - \varepsilon \ge d_{j_0} \checkmark \quad \text{(by how we chose } \varepsilon\text{)}$$

So $\{\tilde{x}_{ij}\}$ is a feasible solution.

Cost of $\{\tilde{x}_{ij}\}$ = cost of $\{x_{ij}\} - c_{1j_0}\varepsilon$

$\Rightarrow \{\tilde{x}_{ij}\}$ has lower cost

$\Rightarrow \{x_{ij}\}$ wasn't optimal!

This proves that if $\{x_{ij}\}$ is optimal then there is no slack in the demand constraints. //

This means that if total demand < total supply then it suffices to solve

$$\begin{cases} \text{minimize} \quad \sum_i \sum_j c_{ij} x_{ij} \\ \text{subject to} \\ \qquad \sum_j x_{ij} \le s_i \quad \text{for } i = 1 .. n \\ \qquad \sum_i x_{ij} = d_j \quad \text{for } j = 1 .. m \\ x_{ij} \ge 0. \end{cases}$$

Now I'll add n slack variables to this system. I'll call them $x_{1,m+1}, x_{2,m+1} .. x_{n,m+1}$

The LP problem is rewritten as

$$\text{minimize} \quad \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^{m+1} x_{ij} = s_i \qquad i = 1 .. n$$

$$\sum_{i=1}^{n} x_{ij} = d_j \qquad \text{for } j = 1 .. m+1^{*}$$

$$x_{ij} \ge 0 \quad \text{for} \quad i = 1 ... n,\ j = 1 .. m+1$$

* note: $d_{m+1} := S - D = \sum_{i=1}^{n} s_i - \sum_{j=1}^{m} d_j$

Look closely at that objective function...

$$\sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{m+1} c_{ij} x_{ij}$$

if $c_{i,m+1} = 0$ for all i! (cost of shipping from the $i^{\underline{th}}$ factory to the $(m+1)^{\underline{st}}$ warehouse is zero.)

~~Whoah~~ Where did the $(m+1)^{\underline{st}}$ warehouse come from? It's our slack variable in disguise

Also, note that I defined the demand of this new warehouse, d_{m+1}, in such a way that the total supply now equals the total demand (once you include the new warehouse).

In this way, you see that introducing a dummy warehouse and zero transportation costs for it is absolutely the right thing to do if total supply > total demand.

Case 3: if total demand > total supply.

Here, there are no feasible solutions of the original problem:

$$\begin{cases} \text{minimize} & \sum_i \sum_j c_{ij} x_{ij} \\ \text{subject to} & \\ & \sum_j x_{ij} \le s_i \qquad i = 1 \ldots n \\ & \sum_i x_{ij} \ge d_j \qquad j = 1 \ldots m \\ x_{ij} \ge 0 & \end{cases}$$

Why? if $\{x_{ij}\}$ were feasible then we would know

$$\sum_i \sum_j x_{ij} \le \sum_i s_i = \text{total supply}.$$

and similarly

$$\sum_j \sum_i x_{ij} \ge \sum_j d_j = \text{total demand}$$

$$\Rightarrow \quad \sum_j d_j \le \sum_i \sum_j x_{ij} \le \sum_i s_i$$

but this is impossible since the left hand side is <u>greater than</u> the right hand side. Therefore there can be no feasible solutions.

So we see that if total demand exceeds total supply then we'll have to completely change the LP problem

<u>idea 1</u>:

$$\text{minimize} \quad \sum_i \sum_j C_{ij} X_{ij}$$

subject to

$$\sum_j X_{ij} \le S_i \qquad i=1..n$$

$$\sum_i X_{ij} \le d_j \qquad j=1..m$$

$$X_{ij} \ge 0$$

i.e. don't ship more from a factory than the factory produces. and don't ship more to a warehouse than they actually want.

This seems quite reasonable! The problem is, the optimal solution is $X_{ij} = 0 \quad \forall i,j$. lazy workers!

Idea 2:

minimize $\sum_i \sum_j C_{ij} X_{ij}$

subject to

$$\sum_{j=1}^{m} X_{ij} = S_i \qquad i = 1 \dots n$$

$$\sum_{i=1}^{n} X_{ij} \leq d_j \qquad j = 1 \dots m$$

$X_{ij} \geq 0$.

i.e. Make sure each factory ships everything it has. And make sure no warehouse receives more than it needs.

This is a good idea! Note: if I introduce slack variables into the demand inequalities $X_{n+1,1}$ $X_{n+1,2}$ $\dots$ $X_{n+1,m}$

then this will be the same as introducing a dummy factory that is producing the extra items that we need. And this will transform the problem into a supply = demand problem if we set the transport costs to zero

Note: In class, I suggested that we set the transport costs from the dummy factory to be some large value: M \$.

In fact, you can check that the prize M doesn't matter. One way is to compute the solution of the dual problem and see that M doesn't affect the values of the objective row. Another way is to draw loops into the transport tableau and observe how different basic feasible solutions don't care about the cost of M.

(For more info, email me or come to office hours.)

ex:
$$C = \begin{pmatrix} 3 & 2 & 7 & 6 \\ 4 & 9 & 1 & 2 \\ 3 & 3 & 10 & 2 \end{pmatrix} \qquad \vec{s} = \begin{pmatrix} 10 \\ 5 \\ 10 \end{pmatrix} \qquad \vec{d} = \begin{pmatrix} 10 \\ 5 \\ 5 \\ 10 \end{pmatrix}$$

optimal solution is

[3] 5	[2] 5	[7]	[6]	10
[4]	[9]	[1] 5	[2]	5
[3]	[3]	[10]	[2] 10	10
[0] 5	[0] 0	[0]	[0] 0	5
10	5	5	10	

cost = \$50

ex:

$$C = \begin{pmatrix} 2 & 2 & 3 \\ 4 & 3 & 4 \\ 1 & 2 & 1 \end{pmatrix} \qquad \vec{s} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix} \qquad \vec{d} = \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}$$

optimal solution is

$\boxed{2}$	$\boxed{2}$ 5	$\boxed{3}$	5
$\boxed{4}$	$\boxed{3}$ 5	$\boxed{4}$	5
$\boxed{1}$	$\boxed{2}$	$\boxed{1}$ 5	5
$\boxed{0}$ 10	$\boxed{0}$	$\boxed{0}$ 5	15
10	10	10	

cost = \$ 30

ex:

$$C = \begin{pmatrix} 3 & 2 & 4 \\ 7 & 9 & 2 \\ 8 & 8 & 8 \\ 3 & 9 & 9 \end{pmatrix} \qquad \vec{s} = \begin{pmatrix} 6 \\ 5 \\ 5 \\ 5 \end{pmatrix} \qquad \vec{d} = \begin{pmatrix} 20 \\ 20 \\ 20 \end{pmatrix}$$

optimal solution is

$\boxed{3}$	$\boxed{2}$ 5	$\boxed{4}$	5
$\boxed{7}$	$\boxed{9}$	$\boxed{2}$ 5	5
$\boxed{8}$	$\boxed{8}$ 5	$\boxed{8}$	5
$\boxed{3}$ 5	$\boxed{9}$	$\boxed{9}$	5
$\boxed{0}$ 15	$\boxed{0}$ 10	$\boxed{0}$ 15	40
20	20	20	

cost = \$ 75