APM 236  Second Midterm  March 19, 2003  100 points possible

You may not use calculators, cell phones, or PDAs during the exam. Partial
credit is possible. Please read the entire test over before starting. Please put a
box around your solutions so that the grader can find them easily.

Print your name clearly:

Print your student number clearly:

Please sign here:

Problem 1  out of 10
Problem 2  out of 15
Problem 3  out of 20
Problem 4  out of 15
Problem 5  out of 10
Problem 6  out of 15
Problem 7  out of 15
Total  out of 100
1. (10 pt) Use the simplex method to solve the following linear programming problem:

Maximize \( x_1 + 2x_2 \)
subject to
\[
\begin{align*}
    x_1 + x_2 & \leq 3 \\
    x_2 & \leq 4
\end{align*}
\]

where \( x_1, x_2 \geq 0 \).

introduce slack vars \( x_3, x_4 \)

maximize \( x_1 + 2x_2 \)
subject to
\[
\begin{align*}
    x_1 + x_2 + x_3 &= 3 \\
    x_2 + x_4 &= 4
\end{align*}
\]
\( x_i \geq 0 \) \( i = 1 \ldots 4 \)

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<tr>
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<th>( x_1 )</th>
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\( x_2 \) enters, \( x_3 \) departs

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terminates! optimal solution is \( \hat{x} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} \)

obj. function equals 6.
2. (15 pt) Use the simplex method to solve the following linear programming problem:

Maximize \( 2x_2 - x_3 \)

subject to

\[
\begin{align*}
x_1 + x_2 - 3x_3 &= 4 \\
x_2 - 2x_3 + x_4 &= 2
\end{align*}
\]

Where \( x_1, x_2, x_3, x_4 \geq 0 \)

don't need to add slack or artificial variables.

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\( x_2 \) enters, \( x_4 \) leaves

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We can make \( x_3 \) go from 0 to positive and the larger \( x_3 \) gets the larger the obj. function will be. No variable will depart so the problem is unbounded. No optimal solution.
3. (20 pt) Use the simplex method to solve the following linear programming problem:

Minimize \(-x_1 + x_2\)

subject to

\[
\begin{align*}
2x_1 + x_2 & \leq 1 \\
-2x_1 + x_2 & \geq 3 \\
\end{align*}
\]

where \(x_1, x_2 \geq 0\)

add two slack variables \(x_3, x_4\) and the artificial variable \(y\)

**Phase 1**: Maximize \(-y\)

Subject to

\[
\begin{align*}
2x_1 + x_2 + x_3 &= 1 \\
-2x_1 + x_2 - x_4 + y &= 3 \\
x_i &\geq 0 \quad i=1,4, y \geq 0
\end{align*}
\]

Write objective in terms of nonbasic variables \(x_1, x_2, x_4\)

\((c_2) \Rightarrow -2x_1 + x_2 - x_4 - 3 = -y\)

\[
\begin{array}{cccccc|c}
& x_1 & x_2 & x_3 & x_4 & y \\
\hline
x_3 & 2 & 1 & 1 & 0 & 0 & 1 \\
y & -2 & 1 & 0 & -1 & 1 & 3 \\
\hline
2 & -1 & 1 & 0 & 0 & 1 & -3
\end{array}
\]

\(x_2\) enters, \(x_3\) departs

\[
\begin{array}{cccccc|c}
& x_1 & x_2 & x_3 & x_4 & y \\
\hline
x_2 & 2 & 1 & 1 & 0 & 0 & 1 \\
y & -4 & 0 & -1 & -1 & 1 & 2 \\
\hline
4 & 0 & 2 & 0 & 0 & -2
\end{array}
\]

Phase 1 terminates prematurely \(\Rightarrow\) No feasible solutions!
4. (15 pt) Consider the linear programming problem:

Maximize \((3 1 2 4)^T \overline{x}\)

subject to

\[
\begin{pmatrix}
1 & 1 & 1 & -1 \\
-2 & 1 & -1 & -1
\end{pmatrix} \overline{x} = \begin{pmatrix} 6 \\ -9 \end{pmatrix}
\]

where \(\overline{x} \geq \overline{0}\).

You are told that at some point while using the simplex method to solve this problem, the basic variables are \(x_1\) and \(x_3\). Find the simplex tableau at that time. *Do not solve this problem by starting the simplex method from scratch and pivoting until you have basic variables \(x_1\) and \(x_3\).*

**basic variables** \(x_1, x_3\)

\[
B = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \Rightarrow B^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}
\]

\[
B^{-1}A_2 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}
\]

\[
B^{-1}A_1 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}
\]

\[
\begin{array}{c|cccc}
3 & 1 & 2 & 4 \\
\hline 
X_1 & X_2 & X_3 & X_4 \\
\hline 
3 & 1 & -2 & 0 & 2 & 3 \\
2 & 0 & 3 & 1 & -3 & 3 \\
\hline 
0 & -1 & 6 & -4 & 15
\end{array}
\]

\[
\begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix} - 1 = -1
\]

\[
\begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \end{pmatrix} - 4 = -4
\]

\[
\begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 15
\]
5. (10 pt) Consider the following tableau:

\[
\begin{array}{cccccccc}
\hline
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
\hline
x_2 & 2 & 1 & -2 & 0 & 2 & 1/3 & 6 \\
x_4 & 4 & 0 & 1 & 1 & 1 & 1/2 & 5 \\
\hline
-4 & 0 & 1 & 0 & 4 & -2 & 10 \\
\hline
\end{array}
\]

a) You arrived at this tableau while applying the simplex method to solve a linear programming problem in which an objective function is to be maximized. What choice of entering and departing variable should you now take if you want the objective function to increase as much as possible? How much will the objective function increase if you make this choice?

- \( x_1 \) enters, \( x_4 \) dep. \( \Rightarrow \) obj. func. incr. by 5
- \( x_6 \) enters, \( x_4 \) dep. \( \Rightarrow \) obj. func. incr. by 20

*take \( x_6 \) entering & \( x_4 \) departing for largest increase.*

b) You arrived at this tableau while applying the simplex method to solve a linear programming problem in which an objective function is to be minimized. What choice of entering and departing variable should you now take if you want the objective function to decrease as much as possible? How much will the objective function decrease if you make this choice?

- \( x_3 \) enters, \( x_4 \) departs \( \Rightarrow \) obj. func. decr. by 5
- \( x_5 \) enters, \( x_2 \) departs \( \Rightarrow \) obj. func. decr. by 12

*take \( x_5 \) entering & \( x_2 \) departing for largest decrease.*
6. (15 pt) Consider the linear programming problem:

\[
\begin{align*}
\text{Minimize } & -x_1 + x_2 \\
\text{subject to } & \\
& x_1 - x_2 \leq 3 \\
& x_1 + x_2 \geq -1 \\
\Rightarrow & -x_1 + x_2 \geq -3 \\
\text{where } & x_1 \geq 0
\end{align*}
\]

a) What is the dual linear programming problem of the above problem?

\[
\text{primal} \begin{cases}
\text{minimize } & -x_1 + x_2 \\
\text{subj. to } & \\
& -x_1 + x_2 \geq -3 \\
& x_1 + x_2 \geq -1 \\
& x_1 \geq 0
\end{cases}
\]

\[
\text{dual} \begin{cases}
\text{maximize } & -3w_1 - w_2 \\
\text{subj. to } & \\
& -w_1 + w_2 \leq -1 \\
& w_1 + w_2 = 1 \\
& w_1, w_2 \geq 0
\end{cases}
\]
b) The optimal solution of the primal problem is at \((x_1, x_2) = (1, -2)\). Use complementary slackness to find an optimal solution of the dual problem.

Value of obj. function at optimal solution of primal is \(-1 + (-2) = -3\)

\[ \Rightarrow -3W_1 - W_2 = -3 \] at opt. solution of dual.

\[ x_1 \neq 0 \Rightarrow (w')_1 = 0 \Rightarrow \text{no slack in 1st eqn.} \]

\[ x_2 \neq 0 \Rightarrow (w')_2 = 0 \Rightarrow \text{no slack in 2nd eqn of dual. (no rows here!)} \]

\[ \Rightarrow \text{solve} \]

\[ -3W_1 - W_2 = -3 \]

\[ -W_1 + W_2 = -1 \]

\[ W_1 + W_2 = 1 \]

\[ (e2) + (e3) \Rightarrow 2W_2 = 0 \Rightarrow W_2 = 0 \Rightarrow W_1 = 1 \]

obj. funct. = -3 \checkmark

Note: didn't actually need complementary slackness. The duality theorem gave us equations 1. We had equation 3. Could have solved from those two.
7. (15 pt) Consider the (primal) linear programming problem:

\[
\text{Maximize } \mathbf{c}^T \mathbf{z} \\
\text{subject to } \\
A\mathbf{z} \leq \mathbf{b}, \\
\text{and } \mathbf{z} \geq \mathbf{0},
\]

and its dual linear programming problem:

\[
\text{Minimize } \mathbf{b}^T \mathbf{w} \\
\text{subject to } \\
A^T \mathbf{w} \geq \mathbf{c}, \\
\text{and } \mathbf{w} \geq \mathbf{0}.
\]

a) Prove that if \( \mathbf{x}_0 \) is a feasible solution of the primal problem and \( \mathbf{w}_0 \) is a feasible solution of the dual problem then

\[
\mathbf{c}^T \mathbf{x}_0 \leq \mathbf{b}^T \mathbf{w}_0.
\]

\[\textbf{Proof:}\]

Since \( \mathbf{x}_0 \) is feasible, we know \( A\mathbf{x}_0 \leq \mathbf{b} \).

Since \( \mathbf{w}_0 \geq 0 \) we take the dot product to preserve the inequality

\[
\Rightarrow \mathbf{w}_0^T A \mathbf{x}_0 \leq \mathbf{w}_0^T \mathbf{b}. \quad (\star)
\]

Since \( \mathbf{w}_0 \) is feasible, we know \( A^T \mathbf{w}_0 \geq \mathbf{c} \).

Since \( \mathbf{x}_0 \geq 0 \) we take the dot product to preserve the inequality

\[
\Rightarrow \mathbf{x}_0^T A^T \mathbf{w}_0 \geq \mathbf{x}_0^T \mathbf{c} \\
\Rightarrow \mathbf{w}_0^T A \mathbf{x}_0 \geq \mathbf{c}^T \mathbf{x}_0. \quad (\star\star)
\]

Combining \((\star)\) and \((\star\star)\),

\[
\mathbf{c}^T \mathbf{x}_0 \leq \mathbf{w}_0^T A \mathbf{x}_0 \leq \mathbf{w}_0^T \mathbf{b} = \mathbf{b}^T \mathbf{w}_0
\]

\[
\Rightarrow \mathbf{c}^T \mathbf{x}_0 \leq \mathbf{b}^T \mathbf{w}_0. \quad \text{done!}
\]
b) Prove that if \( \tilde{x}_0 \) is a feasible solution of the primal problem and \( \tilde{w}_0 \) is a feasible solution of the dual problem such that
\[
\tilde{c}^T \tilde{x}_0 = \tilde{b}^T \tilde{w}_0
\]
then \( \tilde{x}_0 \) is an optimal solution of the primal problem.

From part a, we know that if \( \tilde{x}_1 \) is a feasible solution of the primal problem then we know \( \tilde{c}^T \tilde{x}_1 \leq \tilde{b}^T \tilde{w}_0 \).

Since we're given that \( \tilde{c}^T \tilde{x}_0 = \tilde{b}^T \tilde{w}_0 \) we therefore know
\[
\tilde{c}^T \tilde{x}_1 \leq \tilde{c}^T \tilde{x}_0.
\]

This shows \( \tilde{c}^T \tilde{x}_0 = \max_{\tilde{x}_1 \text{ feasible}} \tilde{c}^T \tilde{x}_1 \)

\( \Rightarrow \tilde{x}_0 \) is an optimal solution.