<< GREEN ANSWER KEY >>

APM 236 Second Midterm March 19, 2003 100 points possible

You may not use calculators, cell phones, or PDAs during the exam. Partial credit is possible. Please read the entire test over before starting. Please put a box around your solutions so that the grader can find them easily.

Print your name clearly:

Print your student number clearly:

Please sign here:

Problem 1	out of 10
Problem 2	out of 15
Problem 3	out of 20
Problem 4	out of 15
Problem 5	out of 10
Problem 6	out of 15
Problem 7	out of 15
Total	out of 100

1. (10 pt) Use the simplex method to solve the following linear programming problem:

$$\text{Maximize } x_1 + 2x_2$$
$$\text{subject to}$$
$$\begin{aligned} x_1 + x_2 &\le 3 \\ x_2 &\le 4 \end{aligned}$$
$$\text{where } x_1, x_2 \ge 0.$$

introduce slack vars x_3, x_4

maximize $x_1 + 2x_2$
subject to

$$\begin{aligned} x_1 + x_2 + x_3 \quad &= 3 \\ x_2 \quad + x_4 &= 4 \end{aligned}$$

$x_i \ge 0 \quad i = 1 \ldots 4$

	x_1	x_2	x_3	x_4	
x_3	1	1	1	0	3
x_4	0	1	0	1	4
	-1	-2	0	0	0

x_2 enters, x_3 departs

	x_1	x_2	x_3	x_4	
x_2	1	1	1	0	3
x_4	-1	0	-1	1	1
	1	0	2	0	6

terminates! optimal solution is $\vec{x} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}$

obj. function equals 6.

2. (15 pt) Use the simplex method to solve the following linear programming problem:

$$\text{Maximize } 2x_2 - x_3$$
$$\text{subject to}$$

$$\begin{array}{rrrrl} x_1 & +x_2 & -3x_3 & & = 4 \\ & x_2 & -2x_3 & +x_4 & = 2 \end{array}$$

$$\text{Where } x_1, x_2, x_3, x_4 \geq 0$$

don't need to add slack or atificial variables.

	x_1	x_2	x_3	x_4	
x_1	1	1	-3	0	4
x_4	0	1	-2	1	2
	0	-2	1	0	0

x_2 enters, x_4 departs

	x_1	x_2	x_3	x_4	
x_1	1	0	-1	-1	2
x_2	0	1	-2	1	2
	0	0	-3	2	4

we can make x_3 go from 0 to positive and the larger x_3 gets the larger the obj. function will be. No variable will depart so the problem is unbounded. No optimal solution.

3. **(20 pt)** Use the simplex method to solve the following linear programming problem:

$$\text{Minimize } -x_1 + x_2$$
subject to
$$\begin{array}{rrl} 2x_1 & +x_2 & \leq 1 \\ -2x_1 & +x_2 & \geq 3 \end{array}$$
where $x_1, x_2 \geq 0$

add two slack variables x_3 & x_4

and the artificial variable y

phase 1: maximize $-y$

subject to

$$2x_1 + x_2 + x_3 = 1$$
$$-2x_1 + x_2 - x_4 + y = 3$$
$$x_i \geq 0 \quad i = 1..4, \; y \geq 0$$

write obj. function in terms of nonbasic variables x_1, x_2, x_4

$$(e2) \Rightarrow \quad -2x_1 + x_2 - x_4 - 3 = -y$$

	x_1	x_2	x_3	x_4	y	
x_3	2	1	1	0	0	1
y	-2	1	0	-1	1	3
	2	-1	1	0	0	-3

x_2 enters, x_3 departs

	x_1	x_2	x_3	x_4	y	
x_2	2	1	1	0	0	1
y	-4	0	-1	-1	1	2
	4	0	2	0	0	-2

phase 1 terminates prematurely

$\Rightarrow$ No feasible solutions!

4. (15 pt) Consider the linear programming problem:

$$\text{Maximize } (3\ 1\ 2\ 4)^T \vec{x}$$
$$\text{subject to}$$
$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ -2 & 1 & -1 & -1 \end{pmatrix} \vec{x} = \begin{pmatrix} 6 \\ -9 \end{pmatrix}$$
$$\text{where } \vec{x} \geq \vec{0}.$$

You are told that at some point while using the simplex method to solve this problem, the basic variables are x_1 and x_3. Find the simplex tableau at that time. *Do not solve this problem by starting the simplex method from scratch and pivoting until you have basic variables x_1 and x_3.*

basic variables x_1 & x_3

$$\to B = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \Rightarrow B^{-1} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$$

$$B^{-1}A_2 = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$B^{-1}\vec{b} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ -9 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$B^{-1}A_4 = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

		3	1	2	4	
		x_1	x_2	x_3	x_4	
3	x_1	1	-2	0	2	3
2	x_3	0	3	1	-3	3
		0	-1	0	-4	15

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix} - 1 = -1$$

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 15$$

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \end{pmatrix} - 4 = -4$$

5. (10 pt) Consider the following tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_2	2	1	-2	0	2	1/3	6
x_4	4	0	1	1	1	1/2	5
	-4	0	1	0	4	-2	10

a) You arrived at this tableau while applying the simplex method to solve a linear programming problem in which an objective function is to be **maximized**. What choice of entering and departing variable should you now take if you want the objective function to **increase** as much as possible? How much will the objective function increase if you make this choice?

X_1 enters, X_4 dep. $\Rightarrow$ obj. func. incr by 5

X_6 enters, X_4 dep $\Rightarrow$ obj. func incr by 20

take X_6 entering & X_4 departing for largest increase.

b) You arrived at this tableau while applying the simplex method to solve a linear programming problem in which an objective function is to be **minimized**. What choice of entering and departing variable should you now take if you want the objective function to **decrease** as much as possible? How much will the objective function decrease if you make this choice?

X_3 enters, X_4 departs $\Rightarrow$ obj. func. decr. by 5

X_5 enters, X_2 departs $\Rightarrow$ obj. func. decr. by 12

take X_5 entering & X_2 departing for largest decrease.

6. (15 pt) Consider the linear programming problem:

$$\text{Minimize } -x_1 + x_2$$
subject to

$$x_1 - x_2 \leq 3 \quad \longrightarrow \quad -x_1 + x_2 \geq -3$$
$$x_1 + x_2 \geq -1$$

where $x_1 \geq 0$

a) What is the dual linear programming problem of the above problem?

primal
$$\begin{cases} \text{minimize } -x_1 + x_2 \\ \text{subj. to} \\ \quad -x_1 + x_2 \geq -3 \\ \quad x_1 + x_2 \geq -1 \\ x_1 \geq 0 \end{cases}$$

dual is

$$\begin{cases} \text{maximize } -3w_1 - w_2 \\ \text{subject to} \\ \quad -w_1 + w_2 \leq -1 \\ \quad w_1 + w_2 = 1 \\ w_1, w_2 \geq 0 \end{cases}$$

b) The optimal solution of the primal problem is at $(x_1, x_2) = (1, -2)$. Use complementary slackness to find an optimal solution of the dual problem.

value of obj. function at optimal solution of primal is $-1 + (-2) = -3$

$\Rightarrow \quad -3w_1 - w_2 = -3$ at opt. solution of dual.

$x_1 \neq 0 \Rightarrow (w')_1 = 0 \Rightarrow$ no slack in first eqn.

$x_2 \neq 0 \Rightarrow (w')_2 = 0 \Rightarrow$ no slack in 2nd eqn of dual. (no news there!)

$\Rightarrow$ solve

$$\begin{aligned} -3w_1 - w_2 &= -3 \\ -w_1 + w_2 &= -1 \\ w_1 + w_2 &= 1 \end{aligned}$$

$(e2) + (e3) \Rightarrow 2w_2 = 0 \Rightarrow w_2 = 0 \Rightarrow w_1 = 1$

obj. funct. $= -3$ ✓

Note: didn't actually need complementary slackness. The duality theorem gave us equation 1. We had equation 3. Could have solved from those two.

7. (15 pt) Consider the (primal) linear programming problem:

$$\text{Maximize } \vec{c}^T \vec{x}$$
subject to
$$A\vec{x} \le \vec{b},$$
$$\text{and } \vec{x} \ge \vec{0},$$

and its dual linear programming problem:

$$\text{Minimize } \vec{b}^T \vec{w}$$
subject to
$$A^T\vec{w} \ge \vec{c},$$
$$\text{and } \vec{w} \ge \vec{0}.$$

a) Prove that if $\vec{x}_0$ is a feasible solution of the primal problem and $\vec{w}_0$ is a feasible solution of the dual problem then

$$\vec{c}^T\vec{x}_0 \le \vec{b}^T\vec{w}_0.$$

proof: Since $\vec{X}_0$ is feasible, we know $A\vec{X}_0 \le \vec{b}$.
Since $\vec{W}_0 \ge 0$ we take the dot product & preserve the inequality

$$\Rightarrow \vec{W}_0^T A \vec{X}_0 \le \vec{W}_0^T \vec{b}. \quad (*)$$

Since $\vec{W}_0$ is feasible, we know $A^T\vec{W}_0 \ge \vec{C}$
Since $\vec{X}_0 \ge 0$ we take the dot product & preserve the inequality

$$\Rightarrow \vec{X}_0^T A^T \vec{W}_0 \ge \vec{X}_0^T \vec{c}$$

$$\Rightarrow \vec{W}_0^T A \vec{X}_0 \ge \vec{c}^T \vec{X}_0 \quad (**)$$

Combining $(*)$ & $(**)$,

$$\vec{c}^T\vec{X}_0 \le \vec{W}_0^T A \vec{X}_0 \le \vec{w}_0^T \vec{b} = \vec{b}^T \vec{W}_0$$

$$\Rightarrow \vec{c}^T\vec{X}_0 \le \vec{b}^T\vec{W}_0.$$ done! //

b) Prove that if $\vec{x}_0$ is a feasible solution of the primal problem and $\vec{w}_0$ is a feasible solution of the dual problem such that

$$\vec{c}^T \vec{x}_0 = \vec{b}^T \vec{w}_0$$

then $\vec{x}_0$ is an **optimal** solution of the primal problem.

From part a, we know that if $\vec{x}_1$ is a feasible solution of the primal problem then we know $\vec{c}^T \vec{x}_1 \leq \vec{b}^T \vec{w}_0$.

Since we're given that $\vec{c}^T \vec{x}_0 = \vec{b}^T \vec{w}_0$ we therefore know

$$\vec{c}^T \vec{x}_1 \leq \vec{c}^T \vec{x}_0$$

is true for <u>every</u> feasible solution $\vec{x}_1$ of the primal problem.

This shows $\vec{c}^T \vec{x}_0 = \max\limits_{\vec{x}_1 \text{ feasible}} \vec{c}^T \vec{x}_1$

$\Rightarrow$ $\vec{x}_0$ is an optimal solution.