APM 236 First Midterm (February 5, 2002) 100 points possible

YELLOW ANSWER KEY

You may not use calculators, cell phones, or PDAs during the exam. Partial credit is possible. Please read the entire test over before starting. Please put a box around your solutions so that the grader can find them easily.

Print your name clearly:

Please sign here:

Problem 1	out of 20
Problem 2	out of 15
Problem 3	out of 5
Problem 4	out of 20
Problem 5	out of 20
Problem 6	out of 20
TOTAL	out of 100

1. Dennis Publishing prints two magazines, *Maxim* and *Blender*, each in units of one hundred. Each unit of *Maxim* requires one unit of ink, three units of paper, and four hours of printing press time to print. Each unit of *Blender* requires two units of ink, three units of paper, and five hours of printing press time to print. The firm has twenty units of ink, forty units of paper, and sixty hours of printing press time available. If the profit realised upon sale is \$200 per unit of *Maxim* and \$300 per unit of *Blender*, how many units of each magazine should the firm print so as to maximize profit?

a. (7 pt) Write this real-world problem as a linear programming problem.

x = units of Maxim
y = units of Blender

ink $= 1 \cdot x + 2y$
paper $= 3 \cdot x + 3y$
press time $= 4 \cdot x + 5y$

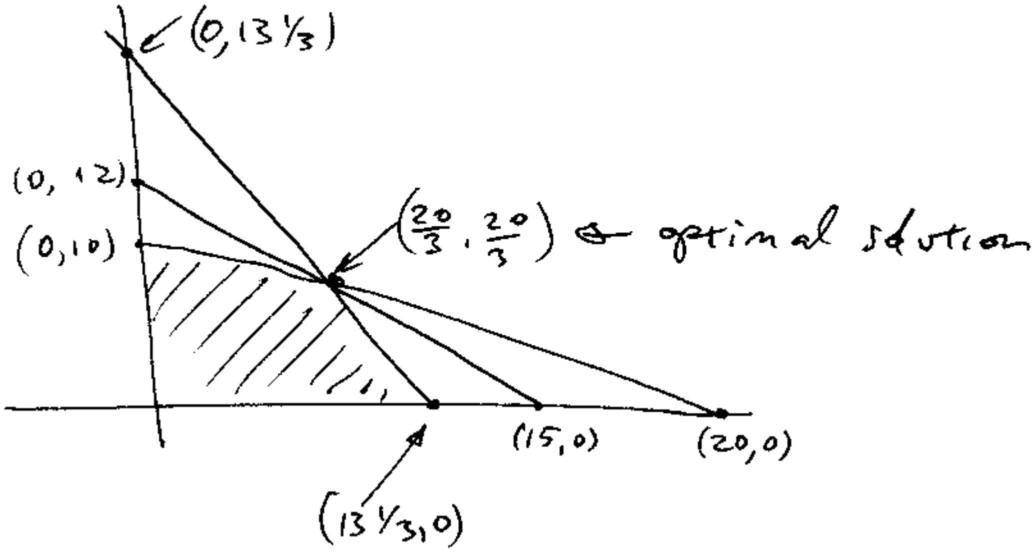
maximize $200x + 300y$
subject to
$$x + 2y \le 20$$
$$3x + 3y \le 40$$
$$4x + 5y \le 60$$
$$x, y \geq 0$$

$x + 2y = 20$, $3x + 3y = 40$ $\rightarrow$ $x + 2y = 20$, $-3y = -20$
$\rightarrow y = \frac{20}{3}$ $x = \frac{20}{3}$

$x + 2y = 20$, $4x + 5y = 60$ $\rightarrow$ $x = \frac{20}{3}$ $y = \frac{20}{3}$

$3x + 3y = 40$, $4x + 5y = 60$ $\rightarrow$ $x = \frac{20}{3}$ $y = \frac{20}{3}$

b. (10 pt) Find the optimal solution(s) of the linear programming problem.



$(0, 10) \to 3{,}000$

$(13\tfrac{1}{3}, 0) \to 8{,}000/3$

$(0, 0) \to 0$

$\left(\frac{20}{3}, \frac{20}{3}\right) \to \frac{10{,}000}{3}$ ← largest!

c. (3 pt) Does your optimal solution(s) use all of the ink available? All of the paper available? All of the printing press time available?

ink $= \frac{20}{3} + 2 \cdot \frac{20}{3} = \frac{60}{3} \stackrel{\checkmark}{=} 20$

paper $= 3 \cdot \frac{20}{3} + 3 \cdot \frac{20}{3} \stackrel{\checkmark}{=} 40$

time $4 \cdot \frac{20}{3} + 5 \cdot \frac{20}{3} \stackrel{\checkmark}{=} 60$

we used exactly all of the paper, ink, and printing press time available.

2\. a. (5 points) Write the following linear programming problem as a linear programming problem in **canonical form**.

Once you have done this, what is your matrix A and your vectors $\vec{c}$ and $\vec{b}$ so that the problem becomes "Maximize $\vec{c} \cdot \vec{x}$ subject to $A\vec{x} = \vec{b}$ and $\vec{x} \geq \vec{0}$?"

$$\begin{aligned} &\text{Maximize } 2x + y \\ &\quad \text{subject to} \\ &\qquad\qquad -x - y \ \leq \ 3 \\ &\qquad\qquad 2x - y \ \geq \ 7 \\ &\quad x, y \geq 0. \end{aligned}$$

$$\begin{cases} \text{maximize } 2x+y \\ \text{subject to} \\ \qquad -x-y+u = 3 \\ \qquad 2x-y -v = 7 \\ x, y, u, v \geq 0 \end{cases}$$

$$A = \begin{pmatrix} -1 & -1 & 1 & 0 \\ 2 & -1 & 0 & -1 \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

$$\vec{c} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

b. (10 points) Write the following linear programming problem in **standard form**.

Once you have done this, what is your matrix A and your vectors $\vec{c}$ and $\vec{b}$ so that the problem becomes "Maximize $\vec{c} \cdot \vec{x}$ subject to $A\vec{x} = \vec{b}$ and $\vec{x} \geq \vec{0}$?"

$$\begin{aligned} &\text{Minimize } -2x + 7y \\ &\text{subject to} \\ &3x - y \leq 4 \\ &x - 8y = 3 \\ &-2x + 7y \geq 1 \longrightarrow 2x - 7y \leq -1 \\ &y \geq 0. \end{aligned}$$

$x = u - v$

$u, v \geq 0$

maximize $2(u-v) - 7y$

subject to

$$\begin{aligned} 3(u-v) - y &\leq 4 \\ u - v - 8y &\leq 3 \\ -u + v + 8y &\leq -3 \\ 2(u-v) - 7y &\leq -1 \end{aligned}$$

$u, v, y \geq 0$

maximize $2u - 2v - 7y$

subject to

$$\begin{aligned} 3u - 3v - y &\leq 4 \\ u - v - 8 &\leq 3 \\ -u + v + 8 &\leq -3 \\ 2u - 2v - 7y &\leq -1 \end{aligned}$$

$u, v, y \geq 0$

$$A = \begin{pmatrix} 3 & -3 & -1 \\ 1 & -1 & -8 \\ -1 & 1 & 8 \\ 2 & -2 & -7 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 4 \\ 3 \\ -3 \\ -1 \end{pmatrix} \quad \vec{c} = \begin{pmatrix} 2 \\ -2 \\ -7 \end{pmatrix}$$

3. (5 points) Find a basic feasible solution of

$$A\vec{x} = \vec{b}$$

where

$$A = \begin{pmatrix} 3 & 0 & 8 & 1 & 0 & -1 \\ 8 & -1 & 0 & 0 & 0 & -2 \\ -2 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\vec{b}^T = (1, 1, 1).$$

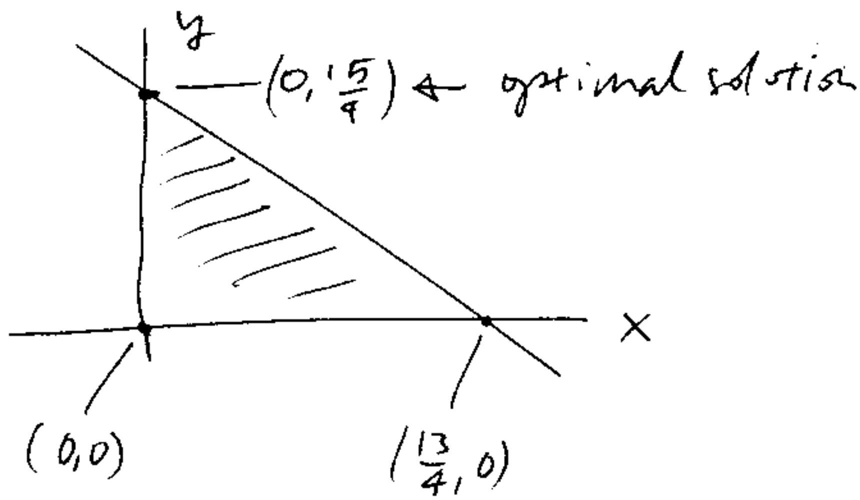
$$\begin{pmatrix} 3 & 0 & 0 \\ 8 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{yields} \quad \begin{pmatrix} 1/3 \\ 1\,2/3 \\ 0 \\ 0 \\ 1\,2/3 \\ 0 \end{pmatrix}$$

in fact, this is the only basic feasible solution.

4. Consider the following linear programming problem:

$$\text{Maximize } 15x + 14y$$
$$\text{subject to}$$
$$\frac{15}{13}x + y \leq \frac{15}{4}$$
$$x, y \geq 0.$$

a. (10 pts) In the above, you are asked to find an optimal solution where x and y are **real numbers** that satisfy the constraints. Use graphical methods to find the optimal solution(s).



x	y	obj. fxn
0	0	0
$\frac{13}{4}$	0	$15 \cdot \frac{13}{4}$
0	$\frac{15}{4}$	$14 \cdot \frac{15}{4}$ ← largest

b. (10 pts) Instead of allowing x and y to be any real number, we now insist that they be **integers** that satisfy the constraints. (I.e., the problem is coming from a real-world problem where non-integer answers make no sense.)

List all integer pairs (x, y) that satisfy the constraints.

x	y	obj fxn
0	0	0
0	1	14
0	2	28
0	3	42
1	0	15
1	1	29
1	2	43
2	0	30
2	1	44
3	0	45 ← optimal!

From this list, find the optimal integer solution(s). Is it the one that you would have expected, based on your answer to the first part of this question?

the optimal integer-valued solution occurs at $(3, 0)$

the optimal real-valued solution occurs at $(0, 15/4)$

they're nowhere near each other!!

Note: this problem really needed a calculator. It would be impossible to do (in reasonable time) without one. :-(

5. In the following, consider linear programming problems with two free variables.

   a. (10 points) Find a set of constraints and an objective function so that the general linear programming problem satisfies the following:

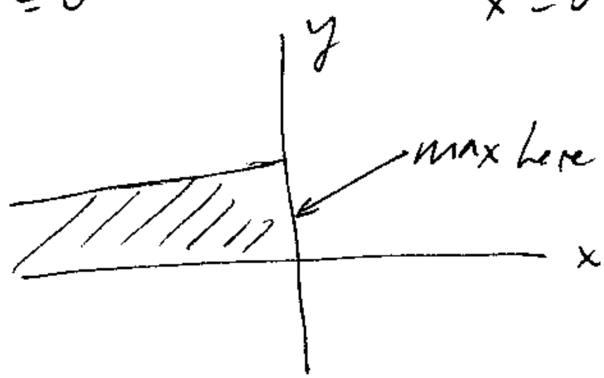
      1) you have an optimal solution(s) if you're trying to **maximize** $\vec{c} \cdot \vec{x}$ over the set of feasible solutions and

      2) you have no optimal solution(s) if you're trying to **minimize** $\vec{c} \cdot \vec{x}$ over the set of feasible solutions.

      Explain your choice graphically.

maximize x
subject to
$y \leq 1$
$y \geq 0$
$x \leq 0$

minimize x
subject to
$y \leq 1$
$y \geq 0$
$x \leq 0$



have max but no min

b. (10 points) Now, keep the same set of constraints (i.e. the set of feasible solutions is unchanged) but choose a *new* objective function $\vec{d} \cdot \vec{x}$ so that

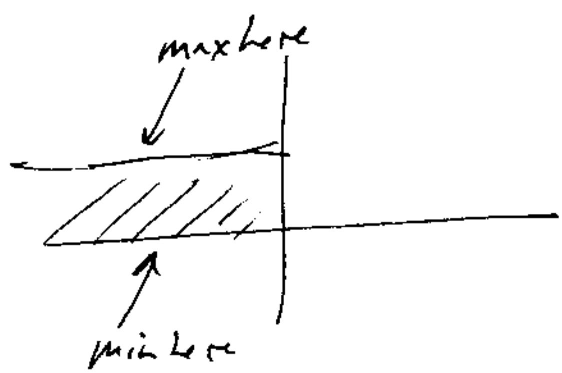
1) you have an optimal solution(s) if you're trying to **maximize** $\vec{d} \cdot \vec{x}$ over the set of feasible solutions and

2) you have an optimal solution(s) if you're trying to **minimize** $\vec{d} \cdot \vec{x}$ over the set of feasible solutions.

Explain your choice graphically.

maximize y
subject to
$y \le 1$
$y \ge 0$
$x \le 0$

minimize y
subject to
$y \le 1$
$y \ge 0$
$x \le 0$



6. Let $\mathcal{S}$ be the set of points that satisfy the constraint

$$A\vec{x} \leq \vec{b}$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

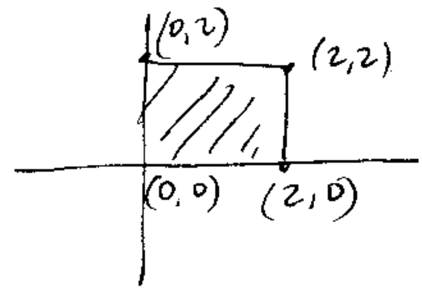
a. (2 pt) Sketch the set $\mathcal{S}$. What are its extreme points?

$x \leq 2$

$y \leq 2$

$-x \leq 0 \rightarrow x \geq 0$

$-y \leq 0 \rightarrow y \geq 0$



extreme points: $(0,0), (2,0), (0,2), (2,2)$

b. (8 pt) Prove that $\mathcal{S}$ is a convex set.

assume $\vec{x} \in S$ and $\vec{y} \in S$. want $t\vec{x} + (1-t)\vec{y} \in S$ for all $t \in (0,1)$

$\vec{x} \in S \Rightarrow A\vec{x} \leq \vec{b} \Rightarrow tA\vec{x} \leq t\vec{b}$ since $t > 0$

$\vec{y} \in S \Rightarrow A\vec{y} \leq \vec{b} \Rightarrow (1-t)A\vec{y} \leq (1-t)\vec{b}$ since $t < 1$

$\Rightarrow tA\vec{x} + (1-t)A\vec{y} \leq t\vec{b} + (1-t)\vec{b} = \vec{b}$

$\Rightarrow A\left(t\vec{x} + (1-t)\vec{y}\right) \leq \vec{b}$

$\Rightarrow t\vec{x} + (1-t)\vec{y} \in S$, as desired.

c. (10 pt) Recall the definition of extreme point: "$\vec{x}$ is an extreme point of $\mathcal{S}$ if there is no pair of distinct points $\vec{y}, \vec{z} \in \mathcal{S}$ such that $\vec{x} = t\vec{y} + (1-t)\vec{z}$ for some $t \in (0,1)$."

Above, you claimed that some collection of points were extreme points of $\mathcal{S}$. Pick one of those points and prove that it really is an extreme point. (*Hint: the easiest way to do this is to assume that you have $\vec{x} = t\vec{y} + (1-t)\vec{z}$ for some $t \in (0,1)$ where $\vec{y} \neq \vec{z}$ and then show that either $\vec{y}$ or $\vec{z}$ is not in $\mathcal{S}$.*)

I claim $(0,0)$ is an extreme point.

assume $(0,0)$ is between $\vec{y}$ and $\vec{z}$ when $\vec{y}$ and $\vec{z} \in S$.

$$\rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} = t\vec{y} + (1-t)\vec{z} \quad \text{for some } t \in (0,1)$$

$$\rightarrow \quad 0 = ty_1 + (1-t)z_1$$
$$0 = ty_2 + (1-t)z_2.$$

since $\vec{y} \in S$ know $y_1 \geq 0$ and $y_2 \geq 0$

similarly, $z_1 \geq 0$ and $z_2 \geq 0$

$$\Rightarrow \quad 0 = ty_1 + (1-t)z_1$$
$$0 = ty_2 + (1-t)z_2$$

since $t \neq 0$ and $t \neq 1$ we have

$y_1 = z_1 = 0$ (since both $y_1, z_1 \geq 0$)

$y_2 = z_2 = 0$ (since both $y_2, z_2 \geq 0$)

$\Rightarrow \vec{y} = \vec{z} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. $\Rightarrow$ cannot find a segment between 2 pts of S that has $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ on the interior of the segment. $\Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is extremal. //