APM 236 First Midterm (February 5, 2002) 100 points possible

WHITE ANSWER KEY

You may not use calculators, cell phones, or PDAs during the exam. Partial credit is possible. Please read the entire test over before starting. Please put a box around your solutions so that the grader can find them easily.

Print your name clearly:

Please sign here:

Problem 1	out of 20
Problem 2	out of 15
Problem 3	out of 5
Problem 4	out of 20
Problem 5	out of 20
Problem 6	out of 20
TOTAL	out of 100

1. A furniture factory owns two lumber operations. The first lumber operation produces half a ton of usable walnut, one ton of usable oak, and one ton of usable pine per day. The second lumber operation produces one ton of usable walnut, one ton of usable oak, and half a ton of usable pine per day. The factory requires at least ten tons of walnut, fifteen tons of oak, and ten tons of pine. If it costs $300 per day to run the first lumber operation and $350 per day to run the second lumber operation, how many days should each operation be run so as to minimize cost?

a. (7 pt) Write this real-world problem as a linear programming problem.

x = # of days of fact 1
y = # of days of fact 2.

walnut = $\frac{1}{2}x + y \geq 10$
oak = $1 \cdot x + y \geq 15$
pine = $1 \cdot x + \frac{1}{2}y \geq 10$

$$\begin{cases} \text{Minimize } 300x + 350y \\ \text{Subject to} \\ \quad \frac{1}{2}x + y \geq 10 \\ \quad x + y \geq 15 \\ \quad x + \frac{1}{2}y \geq 10 \\ x, y \geq 0. \end{cases}$$

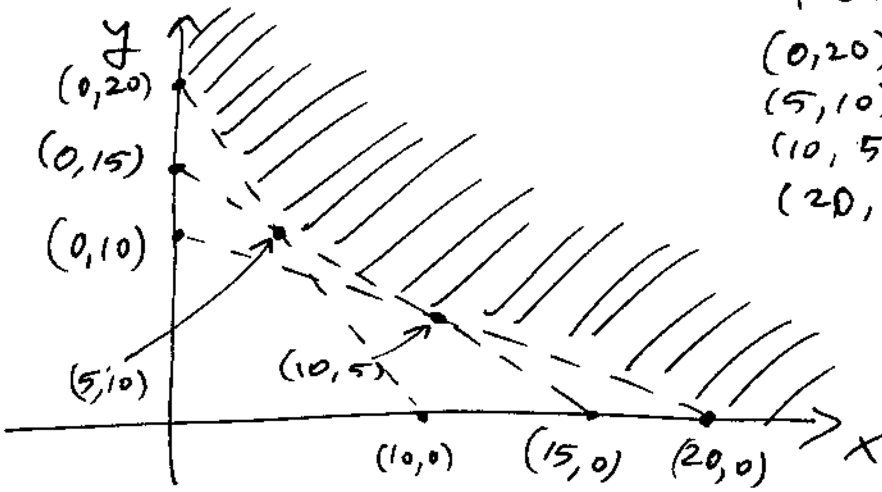
(*)

$300 \cdot 5 + 350 \cdot 10 = 1500 + 3500$

$300 \cdot 10 + 350 \cdot 5 = 3000 + 1750 =$

$$\begin{array}{r} 350 \\ 5 \\ \hline 1750 \end{array}$$

b. (10 pt) Find the optimal solution(s) of the linear programming problem.



4 extreme points

$(0,20) \Rightarrow 7{,}000$

$(5,10) \Rightarrow \quad 5{,}000$

$(10,5) \Rightarrow 4{,}750$

$(20,0) \Rightarrow 6{,}000$

$\frac{1}{2}x + y = 10$ & $x + y = 15 \Rightarrow y = 15 - x \Rightarrow \frac{1}{2}x + (15 - x) = 10$

$\Rightarrow x = 10,\ y = 5$

$x + y = 15$ & $x + \frac{1}{2}y = 10 \Rightarrow y = 15 - x \Rightarrow x + \frac{15}{2} - \frac{x}{2} = 10$

$\Rightarrow \frac{x}{2} = -\frac{5}{2} \Rightarrow x = 5$

$y = 10$

c. (3 pt) Does your optimal solution(s) provide more walnut than needed? More oak than needed? More pine than needed?

optimal soln: $x = 10 \quad y = 5$

walnut produced $= 5 + 5 = 10$

oak produced $= 10 + 5 = 15$

pine produced $= 10 + \frac{5}{2} > 10$

produced just enough walnut & oak

produced too much pine

2. a. (5 points) Write the following linear programming problem as a linear programming problem in **standard form**.

Once you have done this, what is your matrix A and your vectors $\vec{c}$ and $\vec{b}$ so that the problem becomes "Maximize $\vec{c} \cdot \vec{x}$ subject to $A\vec{x} \leq \vec{b}$ and $\vec{x} \geq \vec{0}$?"

$$\begin{aligned} &\text{Maximize } x - y \\ &\text{subject to} \\ &\qquad x + y = 2 \\ &\qquad 2x - y = 4 \\ &x, y \geq 0. \end{aligned}$$

maximize $x-y$

Subject to

$$\begin{aligned} x+y &\leq 2 \\ -x-y &\leq -2 \\ 2x-y &\leq 4 \\ -2x+y &\leq -4 \end{aligned}$$

$x, y \geq 0$

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 2 & -1 \\ -2 & 1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 2 \\ -2 \\ 4 \\ -4 \end{pmatrix}$$

$$\vec{c} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

b. (10 points) Write the following linear programming problem in **canonical form**.

Once you have done this, what is your matrix A and your vectors $\vec{c}$ and $\vec{b}$ so that the problem becomes "Maximize $\vec{c} \cdot \vec{x}$ subject to $A\vec{x} = \vec{b}$ and $\vec{x} \geq \vec{0}$?"

$$\begin{aligned} &\text{Minimize } x + y \\ &\text{subject to} \\ &\quad 2x - y = 1 \\ &\quad x + y \geq 4 \\ &\quad -3x + y \leq 2 \\ &x \geq 0. \end{aligned}$$

$y = u - v \quad u, v \geq 0$

Maximize $-x - u + v$
subject to

$$\begin{aligned} 2x - (u - v) &= 1 \\ x + (u - v) - w &= 4 \\ -3x + (u - v) + z &= 2 \end{aligned}$$

$x, u, v, w, z \geq 0$

maximize $-x - u + v$
subject to

$$\begin{aligned} 2x - u + v \phantom{{}-w+z} &= 1 \\ x + u - v - w \phantom{{}+z} &= 4 \\ -3x + u - v \phantom{{}-w} + z &= 2 \end{aligned}$$

$x, u, v, w, z \geq 0$

$$A = \begin{pmatrix} 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 \\ -3 & 1 & -1 & 0 & 1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \quad \vec{c} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

3. (5 points) Find a basic feasible solution of

$$A\vec{x} = \vec{b}$$

where

$$A = \begin{pmatrix} 2 & 0 & 8 & -1 & 0 & -1 \\ 8 & 1 & 0 & 0 & 0 & 9 \\ -2 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\vec{b}^T = (2, 2, 2).$$

cols 2, 4, 5

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$\rightarrow \begin{matrix} -x_4 = 2 \\ x_2 = 2 \\ x_5 = 2 \end{matrix} \rightarrow \begin{matrix} x_2 = 2 \\ x_4 = -2 \\ x_5 = 2 \end{matrix}$$

no!

cols 2, 3, 5

$$\begin{pmatrix} 0 & 8 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$8x_3 = 2 \rightarrow x_3 = 1/4$$

$$x_2 = 2 \rightarrow x_2 = 2$$

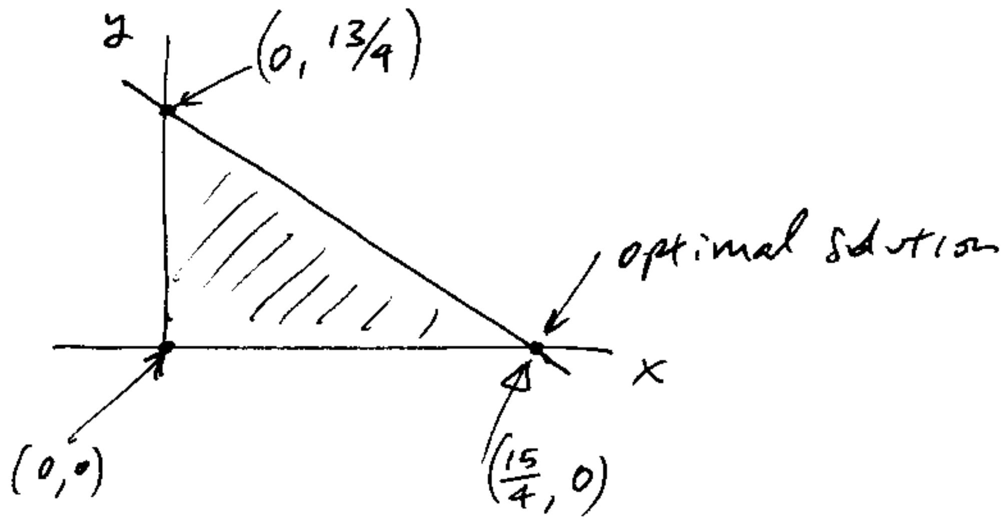
$$-x_3 + x_5 = 2 \rightarrow -1/4 + x_5 = 2 \rightarrow x_5 = 9/4$$

basic feasible soln. $\rightarrow \begin{pmatrix} 0 \\ 2 \\ 1/4 \\ 0 \\ 9/4 \\ 0 \end{pmatrix}$

4. Consider the following linear programming problem:

$$\begin{aligned} &\text{Maximize } 14x + 15y \\ &\quad \text{subject to} \\ &\qquad\qquad x + \frac{15}{13}y \ \leq \ \frac{15}{4} \\ &\quad x, y \geq 0. \end{aligned}$$

a. (10 pts) In the above, you are asked to find an optimal solution where x and y are **real numbers** that satisfy the constraints. Use graphical methods to find the optimal solution(s).



x	y	obj. fxn
0	0	0
15/4	0	$14 \cdot \frac{15}{4}$ ← largest
0	13/4	$15 \cdot \frac{13}{4}$

Note: this problem really needed a calculator. It was impossible to do without one :(

b. (10 pts) Instead of allowing x and y to be any real number, we now insist that they be **integers** that satisfy the constraints. (I.e., the problem is coming from a real-world problem where non-integer answers make no sense.)

List all integer pairs (x, y) that satisfy the constraints.

x	y	Satis. constr?	obj. fxn.
0	0	yes	0
1	0	yes	14
2	0	yes	28
3	0	yes	42
4	0	no	56 x
0	1	yes	15
1	1	yes	29
2	1	yes	43
3	1	no	57 x
0	2	yes	30
1	2	yes	44
2	2	no	58 x

From this list, find the optimal integer solution(s). Is it the one that you would have expected, based on your answer to the first part of this question?

x	y	Satis. constr?	obj. fxn.
0	3	yes	45
1	3	no	59 x

the optimal integer-value solution occurs at $(0,3)$

the optimal real-valued solution occurs at $(\frac{15}{4}, 0)$

they're nowhere near each other!

5. In the following, consider linear programming problems with two free variables.

a. (10 points) Find a set of constraints and an objective function so that the general linear programming problem satisfies the following:

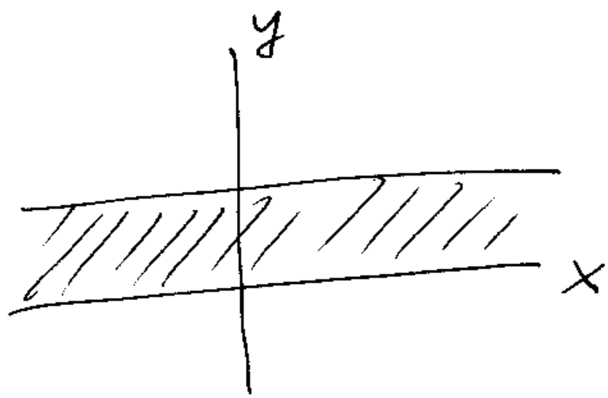
1) you have no optimal solution(s) if you're trying to **maximize** $\vec{c} \cdot \vec{x}$ over the set of feasible solutions and

2) you have no optimal solution(s) if you're trying to **minimize** $\vec{c} \cdot \vec{x}$ over the set of feasible solutions.

Explain your choice graphically.

maximize x
subject to
$y \leq 1$
$y \geq 0$

minimize x
subject to
$y \leq 1$
$y \geq 0$



same feasible region, cannot maximize $\vec{c} \cdot \vec{x}$ and cannot minimize $\vec{c} \cdot \vec{x}$

b. (10 points) Now, keep the same set of constraints (i.e. the set of feasible solutions is unchanged) but choose a *new* objective function $\vec{d} \cdot \vec{x}$ so that

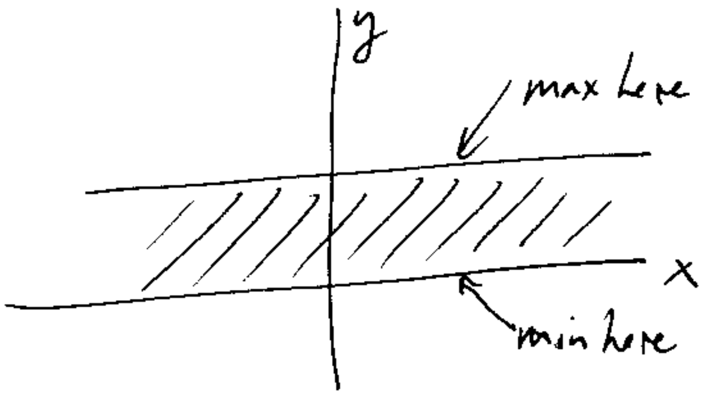
1) you have an optimal solution(s) if you're trying to **maximize** $\vec{d} \cdot \vec{x}$ over the set of feasible solutions and

2) you have an optimal solution(s) if you're trying to **minimize** $\vec{d} \cdot \vec{x}$ over the set of feasible solutions.

Explain your choice graphically.

maximize y
subject to
$y \leq 1$
$y \geq 0$

minimize y
subject to
$y \leq 1$
$y \geq 0$



same set of feasible solutions
but can maximize $\vec{d} \cdot \vec{x}$ and
can minimize $\vec{d} \cdot \vec{x}$.

6. Let S be the set of points that satisfy the constraint

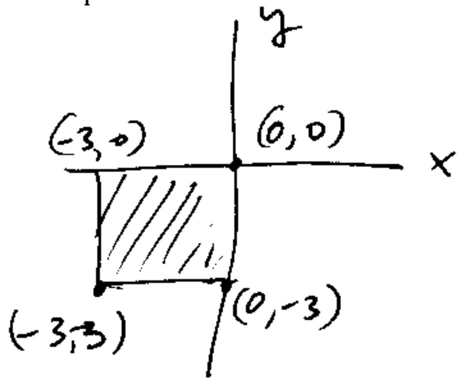
$$A\vec{x} \leq \vec{b}$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 3 \end{pmatrix}$$

a. (2 pt) Sketch the set S. What are its extreme points?

$x \leq 0$
$y \leq 0$
$-x \leq 3 \rightarrow x \geq -3$
$-y \leq 3 \rightarrow y \geq -3$



extremal points:
$(0,0), (0,-3), (-3,0), (-3,-3)$

b. (8 pt) Prove that S is a convex set.

assume $\vec{x}$ and $\vec{y} \in S \Rightarrow A\vec{x} \leq b$ and $A\vec{y} \leq b$.

take $t \in (0,1)$.

then $t > 0 \Rightarrow tA\vec{x} \leq t\vec{b}$

$t < 1 \Rightarrow (1-t)A\vec{y} \leq (1-t)\vec{b}$

summing,

$$tA\vec{x} + (1-t)A\vec{y} \leq t\vec{b} + (1-t)\vec{b} = \vec{b}$$

$$\Rightarrow A\left(t\vec{x} + (1-t)\vec{y}\right) \leq \vec{b}$$

$$\Rightarrow t\vec{x} + (1-t)\vec{y} \in S \Rightarrow S \text{ is convex.}$$

c. (10 pt) Recall the definition of extreme point: "$\vec{x}$ is an extreme point of S if there is no pair of distinct points $\vec{y}, \vec{z} \in S$ such that $\vec{x} = t\vec{y} + (1-t)\vec{z}$ for some $t \in (0,1)$."

Above, you claimed that some collection of points were extreme points of S. Pick one of those points and prove that it really is an extreme point. (*Hint: the easiest way to do this is to assume that you have* $\vec{x} = t\vec{y} + (1-t)\vec{z}$ *for some* $t \in (0,1)$ *where* $\vec{y} \neq \vec{z}$ *and then show that either* $\vec{y}$ *or* $\vec{z}$ *is not in* S.)

I claim $(0,0)$ is an extreme point.
assume $(0,0)$ is between $\vec{y}$ and $\vec{z}$.

$\Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} = t\vec{y} + (1-t)\vec{z}$ for some $t \in (0,1)$

$\rightarrow \quad 0 = ty_1 + (1-t)z_1$
$\quad\; 0 = ty_2 + (1-t)z_2$.

Since $\vec{y} \in S \Rightarrow y_1 \leq 0$ and $y_2 \leq 0$
$\quad\; \vec{z} \in S \Rightarrow z_1 \leq 0$ and $z_2 \leq 0$.

$\Rightarrow \quad 0 = ty_1 + (1-t)z_1$
$\quad\; 0 = ty_2 + (1-t)z_2$

Since $t \neq 0$ and $t \neq 1$, we have
$y_1 = z_1 = 0$ and $y_2 = z_2 = 0$ (since $y_1, y_2 \geq 0$, $z_1, z_2 \geq 0$)

$\Rightarrow \vec{y} = \vec{z} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is extremal since we can't put it between distinct points of S (on a segment)