First HW Problem.

Find the von Neumann value & optimal strategy for each player in the matrix game w/ pay-off matrix

$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{ccccc} q_1 & q_2 & q_3 & q_4 & q_5 \end{array} \\ \begin{pmatrix} -1 & 0 & 2 & -2 & 0 \\ 1 & -2 & -4 & 2 & 2 \\ 0 & -1 & 1 & 1 & -1 \\ 0 & 5 & 4 & 2 & 0 \end{pmatrix} \end{array}$$

row 4 dominates rows 1 & 3 $\Rightarrow$ remove rows 1 & 3

$$\begin{array}{c} \\ p_2 \\ p_4 \end{array} \begin{array}{c} \begin{array}{ccccc} q_1 & q_2 & q_3 & q_4 & q_5 \end{array} \\ \begin{pmatrix} 1 & -2 & -4 & 2 & 2 \\ 0 & 5 & 4 & 2 & 0 \end{pmatrix} \end{array}$$

col 5 dominates col 1 $\Rightarrow$ remove col 5
col 4 dominates col 1 $\Rightarrow$ remove col 4
col 2 dominates col 3 $\Rightarrow$ remove col 2

$$\begin{array}{c} \\ p_2 \\ p_4 \end{array} \begin{array}{c} \begin{array}{cc} q_1 & q_3 \end{array} \\ \begin{pmatrix} 1 & -4 \\ 0 & 4 \end{pmatrix} \end{array}$$

now solve!

player 1 solves:

maximize u

subject to

$$u \le 1 \cdot p_2 + 0 \cdot p_4$$
$$u \le -4p_2 + 4p_4$$
$$p_2 + p_4 = 1$$

$\Rightarrow$ $p_2 = 4/9$, $p_4 = 5/9$, $v = 4/9$

$p_2, p_4 \ge 0$ u unconstrained

player 2 solves:

minimize v

subject to

$$q_1 - 4q_3 \le v$$
$$0q_1 + 4q_3 \le v$$
$$q_1 + q_3 = 1$$

$\Rightarrow$ $q_1 = 8/9$, $q_3 = 1/9$, $v = 4/9$

$q_1, q_3 \ge 0$

player 1 plays $\vec{P} = \begin{pmatrix} 0 \\ 4/9 \\ 0 \\ 5/9 \end{pmatrix}$ player 2 plays $\vec{Q} = \begin{pmatrix} 8/9 \\ 0 \\ 1/9 \\ 0 \\ 0 \end{pmatrix}$

von Neumann value $= 4/9$

2nd HW problem

		II	
		1¢	5¢
I	1¢	2	-5
	5¢	-5	10

there are no dominating rows or columns

Player 1 solves

maximize u

subject to

$$u \leq 2p_1 - 5p_2$$
$$u \leq -5p_1 + 10p_2$$
$$p_1 + p_2 = 1$$
$$p_1, p_2 \geq 0$$

Player 2 solves

minimize v

subject to

$$2q_1 - 5q_2 \leq v$$
$$-5q_1 + 10q_2 \leq v$$
$$q_1 + q_2 = 1$$
$$q_1, q_2 \geq 0$$

$p_1 = 30/44$ $\qquad$ $q_1 = 30/44$

$p_2 = 14/44$ $\qquad$ $q_2 = 14/44$

$u = -10/44$ $\qquad$ $v = -10/44$

$$\vec{P}_* = \vec{Q}_* = \begin{pmatrix} 30/44 \\ 14/44 \end{pmatrix}$$

von Neumann value $= -10/44$

3rd HW problem

		II	
		3♠	4♡
I	2♠	+1	−2
	3♡	−6	+7

No dominating rows or columns.

Player I solves

maximize u
subject to

$$u \le p_1 - 6p_2$$
$$u \le -2p_1 + 7p_2$$
$$p_1 + p_2 = 1$$
$$p_1, p_2 \geq 0$$

Player II solves

minimize v
subject to

$$q_1 - 2q_2 \le v$$
$$-6q_1 + 7q_2 \le v$$
$$q_1 + q_2 = 1$$
$$q_1, q_2 \geq 0$$

opt solns: $\vec{P}_* = \begin{pmatrix} 13/16 \\ 3/16 \end{pmatrix}$ $\quad \vec{Q}_* = \begin{pmatrix} 9/16 \\ 7/16 \end{pmatrix}$ $\quad$ von Neuman value $= -5/16$

Note: since $A \neq A^T$ we're not surprised that $\vec{P}_* \neq \vec{Q}_*$ (as did happen in prev. problem.)

4th HW problem:

Denote a strategy by (L, R) where L = # of pennies in left hand, R = # of pennies in right hand.

		II (0,2)	II (1,1)	II (2,0)
I	(0,2)	0	1	2
	(1,1)	1	0	1
	(2,0)	2	1	0

a) since all entries are ≥ 0, the game is in favor of player I.

b) No dominating rows or columns.

Player 1 solves

max u

subject to

$$u \leq p_2 + 2p_3$$
$$u \leq p_1 + p_3$$
$$u \leq 2p_1 + p_2$$
$$p_1 + p_2 + p_3 = 1$$
$$p_1, p_2, p_3 \geq 0$$

Player 2 solves

min v

subject to

$$q_2 + 2q_3 \leq v$$
$$q_1 + q_3 \leq v$$
$$2q_1 + q_2 \leq v$$
$$q_1 + q_2 + q_3 = 1$$
$$q_1, q_2, q_3 \geq 0$$

an optimal solution for player I is

$$\vec{P}_* = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} \qquad \text{von Neumann value} = \underline{1}$$

an optimal solution for player II is

$$\vec{Q}_* = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} \qquad \text{von Neumann value} = 1$$

(of course)

(again, since $A^T = A$, we have $\vec{Q}_* = \vec{P}_*$).

What other solutions are there? Let's look at perturbing $\vec{Q}_*$ with $\vec{Q}_0$. the new strategy $Q_* + Q_0$ will also be optimal if

$$\vec{P}_*^T A \left(\vec{Q}_* + \vec{Q}_0\right) = 1$$

i.e. $\vec{P}_*^T A \vec{Q}_* + \vec{P}_*^T A \vec{Q}_0 = 1$

Since $\vec{P}_*^T A \vec{Q}_* = 1$ we see that $\vec{Q}_* + \vec{Q}_0$ will also be optimal if $\vec{P}_*^T A \vec{Q}_0 = 0$. Since $\vec{P}_* = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ a short a calculation gives

$$\vec{P}_*^T A \vec{Q}_0 = (\vec{Q}_0)_1 + (\vec{Q}_0)_2 + (\vec{Q}_0)_3 = 0$$

$\Rightarrow$ any perturbation $\vec{Q}_0$ that sums to zero and is small enough so that $\vec{Q}_* + \vec{Q}_0 \geq 0$ will yield another optimal strategy for player II.

e.g. $\vec{Q}_0 = \begin{pmatrix} -1/3 \\ 1/6 \\ 1/6 \end{pmatrix}$ gives $\vec{Q}_* + \vec{Q}_0 = \begin{pmatrix} 1/6 \\ 1/6 \\ 2/3 \end{pmatrix}$

another opt. strategy.

Since $A = A^T$ we can repeat the above argument with perturbations $\vec{P}_0$ of $\vec{P}_*$ and find that player I also has infinitely many ~~str~~ optimal strategies.

c) If player II owes \$100 to player I then (on average) if they play 100 rounds of this game, the debt should be paid off.

d) repeat but with player II having 3 coins

	(0,3)	(1,2)	(2,1)	(3,0)
(0,2)	0	0	1	2
(1,1)	1	0	0	1
(2,0)	2	1	0	0

again, the game is in favor of player 1

Column 4 dominates col 3 ⇒ remove col 4.
col 1 dominates col 2 ⇒ remove col 1

	(1,2)	(2,1)
(0,2)	0	1
(1,1)	0	0
(2,0)	1	0

row 3 dominates row 2 ⇒ remove row 2

	(1,2)	(2,1)
(0,2)	0	1
(2,0)	1	0

Player I solves

$\max u$

subject to

$u \le p_3$

$u \le p_1$

$p_1 + p_3 = 1$

$p_1, p_3 \ge 0$

Player II solves

$\min v$

subject to

$q_3 \le v$

$q_2 \le v$

$q_2 + q_3 = 1$

$q_2, q_3 \ge 0$

optimal solution

$$\vec{P}_* = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} \qquad \vec{Q}_* = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

von Neumann value $= 1/2$

Are there infinitely many strategies for player I?

$$(P_* + P_0)^T A Q_* = P_*^T A Q_* + (P_0)^T A Q_* = 1/2$$

$\Rightarrow$ need $(P_0)^T A Q_* = 0$.

this implies $\frac{1}{2}(P_0)_1 + \frac{1}{2}(P_0)_3 = 0$. $\Rightarrow$ any perturbation $\vec{P}_0$ so that $(\vec{P}_0)_1 + (\vec{P}_0)_3 = 0$ that is small enough so that $\vec{P}_0 + \vec{P}_* \ge 0$ will yield another optimal solution.

Are there infinitely many strategies for player II? Let $\vec{Q}_0$ be a perturbation of $\vec{Q}_*$

$$\vec{P}_*^T A(\vec{Q}_* + \vec{Q}_0) = \vec{P}_*^T A \vec{Q}_* + \vec{P}_*^T A \vec{Q}_0 = 1$$

$$\text{if } \vec{P}_*^T A \vec{Q}_0 = 0.$$

this implies

$$(\vec{Q}_0)_1 + \tfrac{1}{2}(\vec{Q}_0)_2 + \tfrac{1}{2}(\vec{Q}_0)_3 + \tfrac{3}{2}(\vec{Q}_0)_4 = 0.$$

So if $\vec{Q}_0$ is small enough that $\vec{Q}_* + \vec{Q}_0 \geq 0$

and $(\vec{Q}_0)_1 + \frac{1}{2}(\vec{Q}_0)_2 + \frac{1}{2}(\vec{Q}_0)_3 + \frac{3}{2}(\vec{Q}_0)_4 = 0$

and $(\vec{Q}_0)_1 + (\vec{Q}_0)_2 + (\vec{Q}_0)_3 + (\vec{Q}_0)_4 = 0$

then $\vec{Q}_* + \vec{Q}_0$ is another optimal solution.

[<u>Note</u>: the third condition $\Sigma(\vec{Q}_0)_j = 0$ came in because we need that $(\vec{Q}_0) + \vec{Q}_*$ is a mixed strategy. i.e. $\Sigma(\vec{Q}_0 + \vec{Q}_*)_j = 1$. Since $\Sigma(\vec{Q}_*)_j = 1$ already, we need $\Sigma(\vec{Q}_0)_j = 0$.]

For example $\vec{Q}_0 = \begin{pmatrix} 0 \\ 1/6 \\ -1/6 \\ 0 \end{pmatrix}$ works

Player 2 should play (on average) 200 games to repay her debt.

(Note: they're assuming that player II wants to choose a strategy so she can play the game as much as possible before repaying her debt. If she wanted to repay her debt more quickly, she would do this by playing a suboptimal strategy. Or, she could just hand over the \$100 she owes. :))

Fifth HW Problem:

There are three cards in the deck, J, Q, and K.
J < Q < K.

Player I has two options:

Pass: compare cards I either wins \$2 or loses 3\$

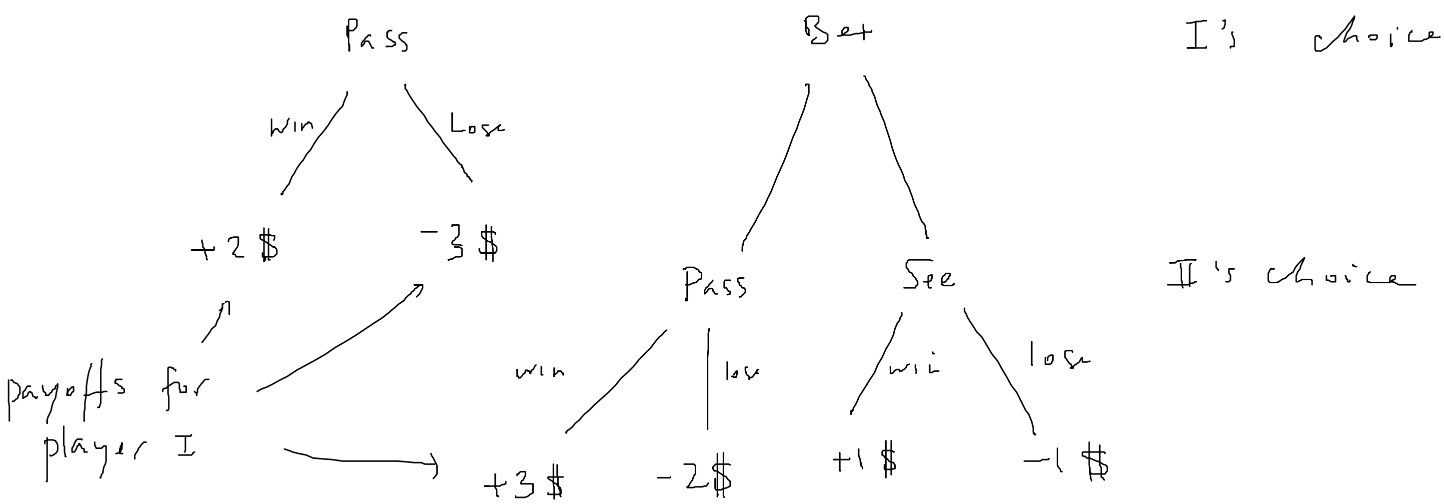
Bet: player I puts \$1 into pot.

If player I bets, player 2 has two options:

Pass: II adds \$3 to pot, I adds \$1 to pot, high hand wins pot (= \$5)

See: I adds \$1 to the pot (now at \$2), high hand wins pot (= \$2)

First, let's make a graph for this game



if player II has a J, she will definitely lose. So she will lose less if she sees rather than if she passes ⇒ II should see on J (and lose only $1)

if player II has K, she will definitely win. So she will win more if she passes rather than sees. ⇒ II should pass on K (and win $2)

if player I has J, she will definitely lose. She's better betting and losing $1 or $2 than passing and losing $3. ⇒ I should bet on J

So player I has 4 strategies

BBB, BBP, BPB, BPP

where BPB means "bet on J, pass on Q, bet on K"

Player II has 2 strategies: SSP and SPP

Now to evaluate the payoff matrix entries

II plays SSP

hand dealt	J Q	J K	Q J	Q K	K J	K Q
probability	1/6	1/6	1/6	1/6	1/6	1/6
player I's result	L	L	W	L	W	W
(BBB)	BS	BP	BS	BP	BS	BS
result: $\boxed{-2/6}$	−1	−2	+1	−2	+1	+1
(BBP)	BS	BP	BS	BP	PS	PS
result: $\boxed{0}$	−1	−2	+1	−2	+2	+2
(BPB)	BS	BP	PS	PP	BS	BS
result: $\boxed{-2/6}$	−1	−2	+2	−3	+1	+1
(BPP)	BS	BP	PS	PP	PS	PS
result: $\boxed{0}$	−1	−2	+2	−3	+2	+2

II plays SPP

hand dealt	J Q	J K	Q J	Q K	K J	K Q
probability	1/6	1/6	1/6	1/6	1/6	1/6
player I's result	L	L	W	L	W	W
(B B B)	B P	B P	B S	B P	B S	B P
result: $\boxed{-1/6}$	−2	−2	+1	−2	+1	+3
(B B P)	B P	B P	B S	B P	P S	P P
result: $\boxed{-1/6}$	−2	−2	+1	−2	+2	+2
(B P B)	B P	B P	P S	P P	B S	B P
result: $\boxed{-1/6}$	−2	−2	+2	−3	+1	+3
(B P P)	B P	B P	P S	P P	P S	P P
result: $\boxed{-1/6}$	−2	−2	+2	−3	+2	+2

$$
\begin{array}{cc}
 & \text{II} \\
 & \begin{array}{cc} (SSP) & (SPP) \end{array} \\
\text{I} \begin{array}{c} (BBB) \\ (BBP) \\ (BPB) \\ (BPP) \end{array} & \begin{pmatrix} -1/3 & -1/6 \\ 0 & -1/6 \\ -1/3 & -1/6 \\ 0 & -1/6 \end{pmatrix}
\end{array}
$$

row 2 dominates row 1 ⇒ remove row 1

row 4 dominates row 3 ⇒ remove row 3

row 4 dominates row 2 ⇒ remove row 2

	(SSP)	(SPP)
(BPP)	0	$-1/6$

column 2 is dominated by column 1
⇒ remove column 1

	(SPP)
(BPP)	$-1/6$

⇒ an optimal strategy for I is the pure strategy: always play (BPP)
an optimal strategy for II is the pure strategy: always play (SPP)
von Neumann value $= -1/6$ \$

Note: this isn't the only optimal solution!

Note: this game was especially tricky to analyze if you're a real card player. Since the ambitious thing for I was to "pass". (In real cards, the ambitious thing is to "bet")

Note: if I had thought a little harder ahead of time, I'm sure I could have found arguments to exclude some of player I's strategies from the beginning.