ANSWER KEY

THE FACULTY OF ARTS AND SCIENCE
University of Toronto

FINAL EXAMINATIONS, APRIL/MAY 2003

APM236H1S
Applications of Linear Programming

Examiner: Professor M. Pugh
Duration: 2 hours

NO AIDS ALLOWED. **Total: 100 marks**

Family Name: ______________________________
(Please Print)

Given Name(s): ______________________________
(Please Print)

Please sign here: ______________________________

Student ID Number: ______________________________

You may not use calculators, cell phones, or PDAs during the exam. Partial credit will be given for partially correct work. Please read through the entire test before starting, and take note of how many points each question is worth. Please put a box around your solutions so that the grader may find them easily.

FOR MARKER'S USE ONLY	
Problem 1:	/10
Problem 2:	/10
Problem 3:	/10
Problem 4:	/30
Problem 5:	/15
Problem 6:	/25
TOTAL:	/100

1. (10 points) Solve the assignment problem where you are trying to **minimize** the objective function

$$\sum_{i=1}^{9}\sum_{j=1}^{9} C_{ij}x_{ij}$$

where the cost matrix C is given below. Show your work. Give the optimal solution and its cost.

Do not solve the problem by inspection! At each step, say what you're doing and make it clear that you're using the Hungarian algorithm of §5.2. To help you, I've provided some copies of the cost matrix so you don't have to recopy it any more than you need to.

$$C = \begin{pmatrix} 0^* & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0^* & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0^* & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0^* & 1 & 1 & 1 & 1 \\ 1 & 1 & 0^* & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0^* & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0^* & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0^* \end{pmatrix}$$

Need to fix row 6 for a complete assignment.

0 in (6,3) to 0* in (5,3) ⇒ col. 3 is necessary.

0 in (6,6) to 0* in (3,6) to 0 in (3,7)

to 0* in (8,7) to 0 in (8,8)

odd length chain ⇒ I can flip it!

$$C = \begin{pmatrix} 0^* & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0^* & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0^* & 1 & 1 \\ 1 & 1 & 1 & 1 & 0^* & 1 & 1 & 1 & 1 \\ 1 & 1 & 0^* & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0^* & 1 & 1 & 1 \\ 1 & 0^* & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0^* & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0^* \end{pmatrix}$$

Extra page if needed.

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned} x_{1\,\underline{1}} &= 1 \\ x_{2\,\underline{4}} &= 1 \\ x_{3\,\underline{7}} &= 1 \\ x_{4\,\underline{5}} &= 1 \\ x_{5\,\underline{3}} &= 1 \\ x_{6\,\underline{6}} &= 1 \\ x_{7\,\underline{2}} &= 1 \\ x_{8\,\underline{8}} &= 1 \\ x_{9\,\underline{9}} &= 1 \\ \text{all other } x_{ij} &= 0 \\ \text{cost } &= 0 \end{aligned}$$

2. (10 points) Solve the assignment problem where you are trying to **maximize** the objective function

$$\sum_{i=1}^{4}\sum_{j=1}^{4} C_{ij}x_{ij}$$

where the cost matrix C is given below. Show your work. Give the optimal solution and its cost.

Do not solve the problem by inspection! At each step, say what you're doing and make it clear that you're using the Hungarian algorithm of §5.2.

$$C = \begin{pmatrix} 10 & 6 & 6 & 9 \\ 9 & 7 & 8 & 11 \\ 7 & 7 & 9 & 5 \\ 10 & 9 & 8 & 10 \end{pmatrix}$$

Since it's a maximization problem, I want to solve the minimization problem with

$$C = \begin{pmatrix} -10 & -6 & -6 & -9 \\ -9 & -7 & -8 & -11 \\ -7 & -7 & -9 & -5 \\ -10 & -9 & -8 & -10 \end{pmatrix}$$

$$\Rightarrow C' = \begin{pmatrix} 0^* & 3 & 4 & 1 \\ 2 & 3 & 3 & 0^* \\ 2 & 1 & 0^* & 4 \\ 0 & 0^* & 2 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} x_{11}=1,\ x_{24}=1, \\ x_{33}=1,\ x_{42}=1 \end{array}$$

cost $= 10+11+9+9 = 39$

3. Consider the transportation problem with cost matrix

$$C = \begin{pmatrix} 3 & 4 & 5 & 4 \\ 6 & 5 & 5 & 6 \\ 6 & 7 & 8 & 4 \end{pmatrix}$$

with supply and demand

$$s_1 = 10, \quad s_2 = 10, \quad s_3 = 20, \qquad d_1 = 5, \quad d_2 = 15, \quad d_3 = 10, \quad d_4 = 10.$$

Your goal is to ship the widgets from the three factories to the four warehouses in the cheapest manner possible.

a. (2 points) Find a basic feasible solution which has the basic variables: x_{11}, x_{12}, x_{23}, x_{24}, x_{32}, and x_{34}.

Cost = $\$195$

$\boxed{3}$ 5	$\boxed{4}$ 5	$\boxed{5}$	$\boxed{4}$	10
$\boxed{6}$	$\boxed{5}$	$\boxed{5}$ 10	$\boxed{6}$ 0	10
$\boxed{6}$	$\boxed{7}$ 10	$\boxed{8}$	$\boxed{4}$ 10	20
5	15	10	10	

b. (8 points) Using this as your initial basic feasible solution, solve the transport problem using transport tableaux.

$$V_1 + W_1 = 3$$
$$V_1 + W_2 = 4$$
$$V_2 + W_3 = 5$$
$$V_2 + W_4 = 6$$
$$V_3 + W_2 = 7$$
$$V_3 + W_4 = 4$$

$\rightarrow$

$V_1 = 0$ $\quad W_1 = 3$

$V_2 = 5$ $\quad W_2 = 4$

$V_3 = 3$ $\quad W_3 = 0$

$W_4 = 1$

$(obj)_{13} = 0+0-5<0$

$(obj)_{14} = 0+1-4=-3<0$

$(obj)_{21} = 5+3-6=2$

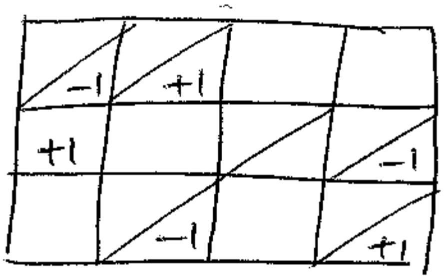
$(obj)_{22} = 5+4-5=4$

$(obj)_{31} = 3+3-6=0$

$(obj)_{33} = 3+0-8<0$

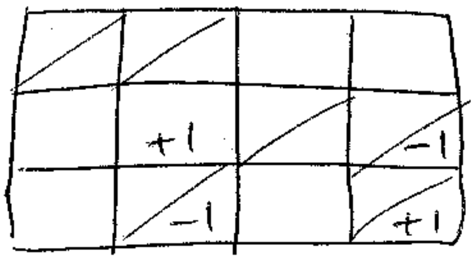
Consider x_{21} and x_{22} as possible entering basic variables.

x_{21} enters?



since $x_{24}=0$ there are no items we can transfer to x_{21}!

x_{22} enters?



since $x_{24}=0$ there are no items we can transfer to x_{22}!

our solution is optimal

4. *This problem runs from page 8 to page 14, in case you would like to read all of it before starting.*

You're the manager of a bike shop and you have three employees: Bob, Rob, and Robert. The store is open five days a week and each day there are two shifts: the morning shift and the evening shift.

Bob hates getting up early. He demands \$160 to work the morning shift and \$80 to work the evening shift. Rob is a flexible fellow and will work either shift for \$120. Robert likes to prowl the bars at night and wants his evenings free. He demands \$100 to work the morning shift and \$200 to work the evening shift. That is, the cost matrix for these workers is:

$$C = \begin{pmatrix} 160 & 80 \\ 120 & 120 \\ 100 & 200 \end{pmatrix}$$

As manager, your job is to assign the shifts to the workers in the cheapest manner possible.

a. By inspection, you can make a good guess at what the optimal solution is. What is your guess? What is its cost?

hire Bob for evenings $x_{12}=1$
hire Robert for mornings $x_{31}=1$
fire Rob. :(cost = \$ 180

b. (20 points) Since there are more workers than shifts, one option is to let them work part-time. This means you require that each shift is fully covered, but you don't require that each worker is fully employed. Write down the linear programming problem that the manager has to solve and use the simplex method to solve it. (*Make sure to say what the optimal solution is and what its cost is!*)

Hint 1: Use your expectations from part a) to guide you in your choice of entering variables. This will reduce the number of simplex tableaux you have to work through.

Hint 2: When you finish phase 1 and start phase 2, don't forget that you're solving a **minimization** problem!

option 1: use x_{ij} to represent the degree of employment
$x_{11}=.2$ → Bob works 1 morn
$x_{12}=.4$ → Bob works 2 eves.

option 2: use x_{ij} to represent the # of shifts being worked
$x_{21}=3$ → Rob works 3 morns
$x_{22}=1$ → Rob works 1 eve.

$\boxed{\text{I chose option 1}}$

Extra page if needed.

minimize $160x_{11} + 80x_{12} + 120x_{21} + 120x_{22} + 100x_{31} + 200x_{32}$

subject to

$x_{11} + x_{12} \le 1$ ← Bob doesn't work more than full time

$x_{21} + x_{22} \le 1$ ← Rob doesn't " "

$x_{31} + x_{32} \le 1$ ← Robert doesn't " "

$x_{11} + x_{21} + x_{31} = 1$ ← the morning shift is covered

$x_{12} + x_{22} + x_{32} = 1$ ← the evening shift is covered.

$x_{ij} \ge 0$

Introduce 3 slack variables y_1, y_2, y_3 and two artificial variables

$x_{11} + x_{12} + y_1 = 1$

$x_{21} + x_{22} + y_2 = 1$

$x_{31} + x_{32} + y_3 = 1$

$x_{11} + x_{21} + x_{31} + z_1 = 1$

$x_{12} + x_{22} + x_{32} + z_2 = 1$

Extra page if needed.

phase 1: maximize $-z_1 - z_2$.

use equations 4 & 5 to write this in terms of nonbasic variables

$$-z_1 - z_2 = x_{11} + x_{21} + x_{31} - 1 + x_{12} + x_{22} + x_{32} - 1$$

	x_{11}	x_{12}	x_{21}	x_{22}	x_{31}	x_{32}	y_1	y_2	y_3	z_1	z_2	
y_1	1	1	0	0	0	0	1	0	0	0	0	1
y_2	0	0	1	1	0	0	0	1	0	0	0	1
y_3	0	0	0	0	1	1	0	0	1	0	0	1
z_1	1	0	1	0	1	0	0	0	0	1	0	1
$\leftarrow z_2$	0	1	0	1	0	1	0	0	0	0	1	1
	-1	-1	-1	-1	-1	-1	0	0	0	0	0	-2

I expect x_{12} and x_{31} to be basic variables for the optimal solution (from part a) so I'll take them as entering variables, if I can.

x_{12} enters, z_2 departs.

Extra page if needed.

	x_{11}	x_{12}	x_{21}	x_{22}	x_{31}	x_{32}	y_1	y_2	y_3	z_1	z_2	
y_1	1	0	0	−1	0	−1	1	0	0	0	−1	0
y_2	0	0	1	1	0	0	0	1	0	0	0	1
y_3	0	0	0	0	1	1	0	0	1	0	0	1
z_1	1	0	1	0	1	0	0	0	0	1	0	1
x_{12}	0	1	0	1	0	1	0	0	0	0	1	1
	−1	0	−1	0	−1	0	0	0	0	0	1	−1

x_{31} enters, z_1 departs.

	x_{11}	x_{12}	x_{21}	x_{22}	x_{31}	x_{32}	y_1	y_2	y_3	z_1	z_2	
y_1	1	0	0	−1	0	−1	1	0	0	0	−1	0
y_2	0	0	1	1	0	0	0	1	0	0	0	1
y_3	−1	0	−1	0	0	1	0	0	1	−1	0	0
x_{31}	1	0	1	0	1	0	0	0	0	1	0	1
x_{12}	0	1	0	1	0	1	0	0	0	0	1	1
	0	0	0	0	0	0	0	0	0	1	1	0

phase 1 terminates!

$x_{12} = 1 \quad x_{31} = 1$

$y_1 = 0$ (Bob fully employed)

$y_2 = 1$ (Rob free)

$y_3 = 0$ (Robert fully employed)

We hope this is our optimal solution since it is what we predicted from part a)

to start phase 2, we need the objective function in terms of the nonbasic variables.

eqn 4 $\Rightarrow$ $X_{31} = -X_{11} - X_{21} + 1$
eqn 5 $\Rightarrow$ $X_{12} = -X_{22} - X_{32} + 1$

objective function $= 160X_{11} + 80(-X_{22} - X_{32} + 1)$
$+ 120X_{21} + 120X_{22}$
$+ 100(-X_{11} - X_{21} + 1) + 200X_{32}$

$= 60X_{11} + 20X_{21} + 40X_{22} + 120X_{32} + 180$

this is what we're trying to minimize. $\Rightarrow$ we're maximizing $-60X_{11} - 20X_{21} - 40X_{22} - 120X_{32} - 180$

$\Rightarrow$ the new objective row will be

$$60 \quad 0 \quad 20 \quad 40 \quad 0 \quad 120 \quad 0 \quad 0 \quad 0 \;\Big|\; 180$$

all ≥ 0 $\Rightarrow$ our solution is optimal.

c. (10 points) Since there are more workers than shifts, another option is to fire one of the workers. This means you require that each shift is fully covered and you require that each worker is fully employed. Write down the linear programming problem that the manager has to solve and solve it either by transportation problem methods or by assignment problem methods. (*Make sure to say what the optimal solution is and what its cost is!*)

Hint: If you're using a transport problem approach, use your expectation from part a) of this question to guide you in your choice of the initial basic feasible solution.

Since supply exceeds demand, we create a fictitious job.

	morn.	eve.	fake		
	$\boxed{160}$	$\boxed{80}$ 1	$\boxed{0}$	1	Bob
	$\boxed{120}$	$\boxed{120}$ 0	$\boxed{0}$ 1	1	Rob
	$\boxed{100}$ 1	$\boxed{200}$	$\boxed{0}$ 0	1	Robert
	1	1	1		

cost = $180

and use our guess from part a) to make a good first guess.

$$V_1 + W_2 = 80$$
$$V_2 + W_2 = 120$$
$$V_2 + W_3 = 0$$
$$V_3 + W_1 = 100$$
$$V_3 + W_3 = 0$$

$$\Rightarrow \quad V_1 = 0 \qquad W_1 = 60$$
$$V_2 = 40 \qquad W_2 = 80$$
$$V_3 = 40 \qquad W_3 = -40$$

Extra page if needed.

$$(obj)_{11} = 0 + 60 - 160 = -100 < 0$$

$$(obj)_{13} = 0 - 40 - 0 = -40 < 0$$

$$(obj)_{21} = 40 + 60 - 120 = -20 < 0$$

$$(obj)_{32} = 40 + 80 - 200 = -80 < 0$$

$\Rightarrow$ our solution is optimal!

Bob $\rightarrow$ evenings

Rob $\rightarrow$ fired

Robert $\rightarrow$ morning

cost = \$ 180

5. *This problem runs from page 15 to page 16, in case you would like to read all of it before starting.*

Scissors-Paper-Stone This is a traditional game. Two players simultaneously name one of three objects: scissors, paper, and stone. If both name the same object, the game is a draw. Otherwise, Scissors cuts Paper, Paper wraps Stone, and Stone breaks Scissors. The player wtih the superior choice (Scissors better than Paper, Paper better than Stone, Stone better than Scissors) wins one dollar from the other player.

a. (5 points) Find the payoff matrix for this game (payoff given in terms of the row player).

II

		sci	pap	st.
	sci.	0	+1	−1
I	pap.	−1	0	+1
	st.	+1	−1	0

b. (2 points) An optimal mixed strategy for the column player is

$$\vec{Q}_* = (1/3, 1/3, 1/3),$$

and an optimal mixed strategy for the row player is

$$\vec{P}_* = (1/3, 1/3, 1/3),$$

This yields the von Neumann value of 0. Use this information to show that your payoff matrix isn't wrong. (Note: this doesn't prove it's right, but you'll notice if it's wrong!)

$$0 \stackrel{?}{=} \vec{P}_*^T A \vec{Q}_*$$

$$0 \stackrel{?}{=} \left(\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}\right)\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} \tfrac{1}{3} \\ \tfrac{1}{3} \\ \tfrac{1}{3} \end{pmatrix} = (0\ 0\ 0)\begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

$= 0$ ✓

c. (8 points) Prove that there are infinitely many optimal strategies for the column player.

if $\vec{Q}_* + \vec{Q}_0$ is another optimal solution, then we know it has to satisfy 3 things:

1) $\vec{Q}_* + \vec{Q}_0$ is a mixed strategy:

i.e. $\vec{Q}_* + \vec{Q}_0 \geq 0 \Rightarrow (Q_0)_1 \geq -\frac{1}{3}$

$(Q_0)_2 \geq -\frac{1}{3}$ $(Q_0)_3 \geq -\frac{1}{3}$

and $\sum_{i=1}^{3} (Q_* + Q_0)_i = 1 \Rightarrow (Q_0)_1 + (Q_0)_2 + (Q_0)_3 = 0.$

2) $\vec{Q}_* + \vec{Q}_0$ is optimal.

i.e. $\vec{P}_*^T A (\vec{Q}_* + \vec{Q}_0) = 0$

i.e. $\vec{P}_*^T A \vec{Q}_* + \vec{P}_*^T A \vec{Q}_0 = 0$

Since $\vec{P}_*^T A = 0$, we see this is true no matter what $\vec{Q}_0$ is.

$\Rightarrow$ any $\vec{Q}_0$ that satisfies conditions 1) will yield a new optimal solution,

eg. $\begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} + \begin{pmatrix} 1/10 \\ 2/10 \\ -3/10 \end{pmatrix}$ will be optimal too!

6. *This problem runs from page 17 to page 24, in case you would like to read all of it before starting.*

Consider the zero-sum matrix game with pay-off matrix

$$C = \begin{pmatrix} 2 & 3 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & -2 \end{pmatrix}$$

a. (2 points) Use domination methods to find a reduced payoff matrix C'.

row 2 dominates row 3 $\Rightarrow$ remove row 3

$$\begin{array}{c} p_1 \\ p_2 \end{array} \begin{pmatrix} 2 & 3 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

col 2 dominates col 1 $\Rightarrow$ remove col 2

col 1 dominates col 4 $\Rightarrow$ remove col 1

$$\begin{array}{cc} & \begin{array}{cc} q_3 & q_4 \end{array} \\ \begin{array}{c} p_1 \\ p_2 \end{array} & \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \end{array}$$

b. (2 points) What is the linear programming problem that the row player must solve in order to find an optimal mixed strategy?

max u

subj. to

$u \le -p_1 + p_2$

$u \le 2p_1 - p_2$

$p_1 + p_2 = 1$

$p_1, p_2 \ge 0$

c. (1 point) What is the linear programming problem that the column player must solve in order to find an optimal mixed strategy?

min v

subj. to

$-q_3 + 2q_4 \le v$

$q_3 - q_4 \le v$

$q_3 + q_4 = 1$

$q_3, q_4 \ge 0$

d. (15 points) Use the simplex method to find an optimal strategy for the row player. What is the von Neumann value of this game?

max u
subj. to

$p_1 - p_2 + u \le 0$
$-2p_1 + p_2 + u \le 0$
$p_1 + p_2 = 1$

$p_1, p_2 \ge 0$

$\Rightarrow$

max $u - v$
subj to

$p_1 - p_2 + u - v \le 0$
$-2p_1 + p_2 + u - v \le 0$
$p_1 + p_2 = 1$

$p_1, p_2, u, v \ge 0$

introduce 2 slack variables & one artificial variable.

<u>phase 1:</u> max $-z$
Subject to

$p_1 - p_2 + u - v + y_1 = 0$
$-2p_1 + p_2 + u - v + y_2 = 0$
$p_1 + p_2 + z = 1$

$p_1, p_2, u, v, y_1, y_2, z \ge 0.$

Extra page if needed.

the objective row will come from maximizing $p_1 + p_2 - 1 = -z$

	p_1	p_2	u	v	y_1	y_2	z	
y_1	1	-1	1	-1	1	0	0	0
y_2	-2	1	1	-1	0	1	0	0
z	1	1	0	0	0	0	1	1
	-1	-1	0	0	0	0	0	-1

p_1 enters, y_1 departs.

	p_1	p_2	u	v	y_1	y_2	z	
p_1	1	-1	1	-1	1	0	0	0
y_2	0	-1	3	-3	2	1	0	0
z	0	2	-1	1	-1	0	1	1
	0	-2	1	-1	1	0	0	-1

p_2 enters, z departs.

Extra page if needed.

	p_1	p_2	u	v	y_1	y_2	z	
p_1	1	0	½	−½	½	0	½	½
y_2	0	0	5/2	−5/2	3/2	1	½	½
p_2	0	1	−½	½	−½	0	½	½
	0	0	0	0	0	0	1	0

and phase 1 terminates!

now to phase 2.

we want to maximize $u-v$.

we're in luck! our objective function is already in terms of nonbasic variables! :)

	p_1	p_2	u	v	y_1	y_2	
p_1	1	0	½	−½	½	0	½
y_2	0	0	5/2	−5/2	3/2	1	½
p_2	0	1	−½	½	−½	0	½
	0	0	−1	1	0	0	0

u enters, y_2 departs

$\frac{\frac{1}{2}}{\frac{1}{2}} = 1 \quad \frac{\frac{1}{2}}{5/2} = 1/5$

Extra page if needed.

	p_1	p_2	u	v	y_1	y_2	
p_1	1	0	0	0	$1/5$	$-1/5$	$2/5$
u	0	0	1	-1	$3/5$	$2/5$	$1/5$
p_2	0	1	0	0	$-1/5$	$1/5$	$3/5$
	0	0	0	0	$3/5$	$2/5$	$1/5$

and phase 2 terminates!

optimal solution:

$p_1 = 2/5$

$p_2 = 3/5$

$p_3 = 0$

von Neumann value $= 1/5$

e. (5 points) Use complementary slackness to find an optimal strategy for the column player.

max u

subj. to

$$p_1 - p_2 + u \le 0$$
$$-2p_1 + p_2 + u \le 0$$
$$p_1 + p_2 = 1$$
$$p_1, p_2 \ge 0$$

<u>dual</u>:

min v

subj. to

$$q_3 - 2q_4 + v \ge 0$$
$$-q_3 + q_4 + v \ge 0$$
$$q_3 + q_4 = 1$$
$$q_3, q_4 \ge 0$$

$p_1 \ne 0 \Rightarrow$ no slack in ~~eqn 1~~ first ineq.

$p_2 \ne 0 \rightarrow$ no slack in 2nd einequality

solve $v = 1/5$

$$q_3 - 2q_4 + v = 0$$
$$-q_3 + q_4 + v = 0 \Rightarrow$$
$$q_3 + q_4 = 1$$

$$q_3 = 3/5$$
$$q_4 = 2/5$$
$$v = 1/5$$

optimal strategy for column player is

$$Q_* = \begin{pmatrix} 0 \\ 0 \\ 3/5 \\ 2/5 \end{pmatrix}$$

von Neumann value $= 1/5$