

**THE FACULTY OF ARTS AND SCIENCE**  
**University of Toronto**  
**FINAL EXAMINATIONS, APRIL/MAY 2003**

**APM236H1S**  
**Applications of Linear Programming**

Examiner: Professor M. Pugh  
Duration: 2 hours

**NO AIDS ALLOWED.**

**Total: 100 marks**

Family Name: \_\_\_\_\_  
(Please Print)

Given Name(s): \_\_\_\_\_  
(Please Print)

Please sign here: \_\_\_\_\_

Student ID Number: \_\_\_\_\_

**You may not use calculators, cell phones, or PDAs during the exam. Partial credit will be given for partially correct work. Please read through the entire test before starting, and take note of how many points each question is worth. Please put a box around your solutions so that the grader may find them easily.**

FOR MARKER'S USE ONLY	
Problem 1:	/10
Problem 2:	/10
Problem 3:	/10
Problem 4:	/30
Problem 5:	/15
Problem 6:	/25
TOTAL:	/100

1. (10 points) Solve the assignment problem where you are trying to **minimize** the objective function

$$\sum_{i=1}^9 \sum_{j=1}^9 C_{ij}x_{ij}$$

where the cost matrix  $C$  is given below. Show your work. Give the optimal solution and its cost.

*Do not solve the problem by inspection! At each step, say what you're doing and make it clear that you're using the Hungarian algorithm of §5.2. To help you, I've provided some copies of the cost matrix so you don't have to recopy it any more than you need to.*

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Extra page if needed.

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$x_{1\_} = 1$$

$$x_{2\_} = 1$$

$$x_{3\_} = 1$$

$$x_{4\_} = 1$$

$$x_{5\_} = 1$$

$$x_{6\_} = 1$$

$$x_{7\_} = 1$$

$$x_{8\_} = 1$$

$$x_{9\_} = 1$$

$$\text{all other } x_{ij} = 0$$

$$\text{cost} =$$

2. (10 points) Solve the assignment problem where you are trying to **maximize** the objective function

$$\sum_{i=1}^4 \sum_{j=1}^4 C_{ij}x_{ij}$$

where the cost matrix  $C$  is given below. Show your work. Give the optimal solution and its cost.

*Do not solve the problem by inspection! At each step, say what you're doing and make it clear that you're using the Hungarian algorithm of §5.2.*

$$C = \begin{pmatrix} 10 & 6 & 6 & 9 \\ 9 & 7 & 8 & 11 \\ 7 & 7 & 9 & 5 \\ 10 & 9 & 8 & 10 \end{pmatrix}$$

Extra page if needed.



Extra page if needed.

4. *This problem runs from page 8 to page 15, in case you would like to read all of it before starting.*

You're the manager of a bike shop and you have three employees: Bob, Rob, and Robert. The store is open five days a week and each day there are two shifts: the morning shift and the evening shift.

Bob hates getting up early. He demands \$160 to work the morning shift and \$80 to work the evening shift. Rob is a flexible fellow and will work either shift for \$120. Robert likes to prowl the bars at night and wants his evenings free. He demands \$100 to work the morning shift and \$200 to work the evening shift. That is, the cost matrix for these workers is:

$$C = \begin{pmatrix} 160 & 80 \\ 120 & 120 \\ 100 & 200 \end{pmatrix}$$

As manager, your job is to assign the shifts to the workers in the cheapest manner possible.

- a. By inspection, you can make a good guess at what the optimal solution is. What is your guess? What is its cost?

- b. (20 points) Since there are more workers than shifts, one option is to let them work part-time. This means you require that each shift is fully covered, but you don't require that each worker is fully employed. Write down the linear programming problem that the manager has to solve and use the simplex method to solve it. (*Make sure to say what the optimal solution is and what its cost is!*)

Hint 1: Use your expectations from part a) to guide you in your choice of entering variables. This will reduce the number of simplex tableaux you have to work through.

Hint 2: When you finish phase 1 and start phase 2, don't forget that you're solving a **minimization** problem!!

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- c. (10 points) Since there are more workers than shifts, another option is to fire one of the workers. This means you require that each shift is fully covered and you require that each worker is fully employed. Write down the linear programming problem that the manager has to solve and solve it either by transportation problem methods or by assignment problem methods. (*Make sure to say what the optimal solution is and what its cost is!*)

Hint: If you're using a transport problem approach, use your expectation from part a) of this question to guide you in your choice of the initial basic feasible solution.

Extra page if needed.

5. *This problem runs from page 16 to page 17, in case you would like to read all of it before starting.*

**Scissors-Paper-Stone** This is a traditional game. Two players simultaneously name one of three objects: scissors, paper, and stone. If both name the same object, the game is a draw. Otherwise, Scissors cuts Paper, Paper wraps Stone, and Stone breaks Scissors. The player with the superior choice (Scissors better than Paper, Paper better than Stone, Stone better than Scissors) wins one dollar from the other player.

- a. (5 points) Find the payoff matrix for this game (payoff given in terms of the row player).

- b. (2 points) An optimal mixed strategy for the column player is

$$\vec{Q}_* = (1/3, 1/3, 1/3),$$

and an optimal mixed strategy for the row player is

$$\vec{P}_* = (1/3, 1/3, 1/3),$$

This yields the von Neumann value of 0. Use this information to show that your payoff matrix isn't wrong. (Note: this doesn't prove it's right, but you'll notice if it's wrong!)

- c. (8 points) Prove that there are infinitely many optimal strategies for the column player.

6. *This problem runs from page 18 to page 24, in case you would like to read all of it before starting.*

Consider the zero-sum matrix game with pay-off matrix

$$C = \begin{pmatrix} 2 & 3 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & -2 \end{pmatrix}$$

- a. (2 points) Use domination methods to find a reduced payoff matrix  $C'$ .
- b. (2 points) What is the linear programming problem that the row player must solve in order to find an optimal mixed strategy?

- c. (1 point) What is the linear programming problem that the column player must solve in order to find an optimal mixed strategy?
- d. (15 points) Use the simplex method to find an optimal strategy for the row player. What is the von Neumann value of this game?

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- e. (5 points) Use complementary slackness to find an optimal strategy for the column player.