Game Theory!

We will look at two person, zero sum matrix games.

ex: Two players Dr Even & Dr Odd each secretly think of an integer between 1 and 3, inclusive. Both players reveal their numbers simultaneously. If the sum of their numbers is even then Dr Even wins their difference from Dr Odd. (provided the numbers are distinct). If the numbers are equal, then Dr Even wins their sum from Dr Odd. If the sum of the numbers is odd then Dr Even pays \$3 to Dr Odd.

Given this game, does one of the players have an advantage? If so, how much of an advantage? Is there an optimal strategy for each player that will maximize their winnings and minimize their losses?

First, let's make the payoff matrix

$$
\begin{array}{cc}
 & \text{Dr Odd} \\
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\text{Dr Even} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 2 & -3 & 2 \\ -3 & 4 & -3 \\ 2 & -3 & 6 \end{bmatrix}
\end{array}
$$

Note: the entries are done with the perspective of the row player, Dr Even.

for ex. Dr Even chooses 3, Dr Odd chooses 2] sum = 5 ⇒ Dr Odd wins \$3. (Dr Even loses \$3)

Dr Even chooses 2, Dr Odd chooses 2] sum = 4 (even) values equal ⇒ Dr Even wins 2+2

Dr Even chooses 1, Dr Odd chooses 3] sum = 4 (even) values not equal ⇒ Dr Even wins 3−1 = 2.

Looking at all possible 9 plays, we see that we fill in the table.

Now that we have the payoff matrix, can we deduce anything?

Well first of all, we see that given the choice between playing 1 and 3, Dr Even will always do as well or better by playing 3.

Dr Even	Dr Odd	Dr Even wins
1	1	2
3	1	2
1	2	-3
3	2	-3
1	3	2
3	3	6

So Dr Even should never play a 1.
Similarly, Dr Odd should never play 3

So the reduced payoff matrix is

$$
\begin{array}{cc|cc|}
 & & \multicolumn{2}{c}{\text{Dr Odd}} \\
 & & 1 & 2 \\
\hline
\text{Dr Even} & 2 & -3 & 4 \\
 & 3 & 2 & -3 \\
\hline
\end{array}
$$

Here we applied domination. i.e. Whenever a row in a payoff matrix is term-by-term less than or equal to another row, delete the smaller row. (Since the row player is trying to maximize the outcome.)

Whenever a column of a payoff matrix is entry by entry greater than or equal to another column, delete the larger column. (Because the column player is trying to minimize losses.)

Continue deleting rows & columns until there are no dominant rows or columns left.

Q: Okay, we have the game and we have the reduced payoff matrix. How do we analyze it?

Here we run into the most annoying thing about baby game theory. The assumption of a fixed strategy. (More advanced game theory can be trickier.)

i.e. The game is described to you, you think for a while, choose a strategy, and stick with it. In fact, you could explain your strategy to a monkey and Dr Odd would explain her strategy to her monkey, the two of you would leave the room and let your monkeys play game after game. You'd return later to claim your winnings.

This seems a little odd, doesn't it? Let's look at fixed strategies a little more.

To make things worse, not only do you have to keep a single strategy, but you have to tell your opponent ahead of time what your strategy will be.
Given that your opponent will know your strategy what's the best strategy to choose?

(Boy what a strange way to play a game!)

If you are Dr Even and you tell Dr Odd that your strategy will be to always play a 2, then Dr Odd will realize that she can constantly win if she always plays a 1.

Similarly, if you are Dr Odd and you tell Dr Even that your strategy will be to constantly play a 2 then Dr Even will realize that she can constantly win if she always plays a 2.

So you want to choose a strategy that cannot be used against you. Or, if all strategies are going to hurt you, you want to choose the one that causes the least loss.
If there are strategies that will always lead to gain, find the one w/ the most gain.

This is still confusing, isn't it? You're asking yourself who declares their strategy first.

Look at it this way. Dr Even & Dr Odd each choose a strategy and put it face down on the table. A fair coin is flipped and the "winner" gets to look at the "loser's" strategy and choose a new strategy according to a best response. What strategy would you choose?

Or, imagine going into the game where you have to play the same strategy over and over but your opponent is free to observe you and guess what your strategy is and respond to it. If you're Dr Odd and you play only 1's then after a while your opponent will guess that you're fond of 1's and will start to play 3's and you will start to lose all the time. (Even if you are Dr Odd and your strategy is to play 1 90% of the time and 2 10% of the time, your opponent will soon collect enough statistics to guess that you're biased toward 1's and will then start playing 2's all the time. Dr Even will then win 2 90% of the time, lose 3 10% of the time with a net payoff to Dr Even of $2\left(\frac{90}{100}\right) - 3\left(\frac{10}{100}\right) = \$1.50$.

<u>defn</u>: Let A be an $m \times n$ matrix game. A mixed (or probabilistic) strategy for the row player is a column vector

$$\vec{P} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{pmatrix} \quad \text{such that } \vec{P} \geq 0$$

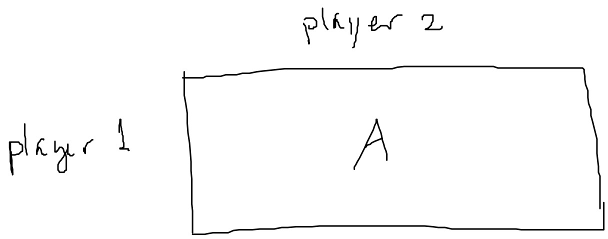
and $\sum_{i=1}^{m} p_i = 1$. A mixed (or probabilistic) strategy for the column player is a row vector $\vec{Q} = (q_1\ q_2\ \ldots\ q_n)$ such that

$$\vec{Q} \geq 0 \quad \text{and} \quad \sum_{j=1}^{n} q_j = 1$$

Any mixed strategy containing an entry of 1 (whence all other entries are necessarily 0) is called a <u>pure strategy</u>.

The interpretation of the mixed strategy for the row player is that she will choose row i of the matrix A with probability p_i

Consider a two-person zero-sum game with payoff matrix A where A is $m \times n$.



player 1 will have a choice of m pure strategies to play. A mixed strategy $\vec{P} \in \mathbb{R}^m$ is a vector $\vec{P} \geq \vec{0}$ so that $\sum_{i=1}^{m} P_i = 1$. It represents: "play pure strategy $\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ with probability P_1, play pure strategy $\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ with probability P_2, etc".

Similarly, player 2 will have a choice of n pure strategies to play. A mixed strategy $\vec{Q} \in \mathbb{R}^n$ is a vector $\vec{Q} \geq \vec{0}$ so that $\sum_{j=1}^{n} Q_j = 1$.

If player 1 plays her i^{th} pure strategy and player 2 plays her j^{th} pure strategy, the pay-off to player 1 is (by definition) A_{ij}.
The loss to player 2 is (since a zero-sum game) $-A_{ij}$

The probability that player 1 plays the ith strategy and that player 2 plays the jth strategy is $p_i q_j$

$\Rightarrow$ the expected contribution to player 1's pocketbook from this pair of strategies is

$$p_i A_{ij} q_j \quad (= p_i q_j A_{ij})$$

Summing over all possible pairs of strategies, the expected winnings of player 1 are:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} p_i A_{ij} q_j = E(\vec{P}, \vec{Q}) = \vec{P}^T A \vec{Q}$$

So player 1 would like to choose $\vec{P}$ so that whatever $\vec{Q}$ is, $E(\vec{P}, \vec{Q})$ is as <u>big</u> as possible.

Similarly, player 2 would like to choose $\vec{Q}$ so that whatever $\vec{P}$ is, $E(\vec{P}, \vec{Q})$ is as <u>small</u> as possible

Suppose player 1 chooses a strategy $\vec{P}$. Then she can expect to receive <u>at least</u>

$$\min_{\vec{Q}} E(\vec{P}, \vec{Q}) = \min_{\vec{Q}} \vec{P}^T A Q$$

So player 1 would like to find $\vec{P}$ that maximizes this worst-case scenario.

Player 1 goal: find $\vec{P}_*$ that maximizes

$$\min_{\vec{Q}} \vec{P}^T A \vec{Q}. \text{ i.e. solve}$$

$$\max_{\vec{P}} \min_{\vec{Q}} \vec{P}^T A \vec{Q}.$$

Similarly, suppose player chooses a strategy $\vec{Q}$ then she can expect to lose no more than

$$\max_{\vec{P}} \vec{P}^T A \vec{Q}.$$

She wants to find $\vec{Q}$ that minimizes this worst case scenario.

Player 2 goal: find $\vec{Q}_*$ that minimizes

$$\max_{\vec{P}} \vec{P}^T A \vec{Q} \text{ i.e. solve}$$

$$\min_{\vec{Q}} \max_{\vec{P}} \vec{P}^T A \vec{Q}.$$

$\Rightarrow$ player 1 wants to solve

$$\max_{\vec{P}} \min_{\vec{Q}} \sum_{i=1}^{m} \sum_{j=1}^{n} P_i A_{ij} q_j = \max_{\vec{P}} \min_{\vec{Q}} \sum_{i=1}^{n} P_i \sum_{j=1}^{n} A_{ij} Q_j$$

player 2 wants to solve

$$\min_{\vec{Q}} \max_{\vec{P}} \sum_{i=1}^{m} \sum_{j=1}^{n} P_i A_{ij} q_j = \min_{\vec{Q}} \max_{\vec{P}} \sum_{j=1}^{n} q_j \sum_{i=1}^{m} A_{ij} P_i$$

What's an easier way to understand this?

<u>lemma</u>: Let $\{\phi_i\}_1^n$ and $\{\psi_j\}_1^n$ be sequences of numbers. where $\vec{P} \geq 0$ and $\sum_{i=1}^{m} P_i = 1$

and $\vec{Q} \geq 0$ and $\sum_{j=1}^{n} q_j = 1$

then
$$\max_{\vec{P}} \sum_{i=1}^{m} \phi_i P_i = \max_i \phi_i$$

and
$$\min_{\vec{Q}} \sum_{j=1}^{n} \psi_j q_j = \min_j \psi_j$$

This lemma implies

$$\min_{\vec{Q}} \max_{\vec{P}} \sum_{i=1}^{n} P_i \sum_{j=1}^{n} A_{ij} Q_j = \min_{\vec{Q}} \max_{i=1..m} \sum_{j=1}^{n} A_{ij} Q_j$$

$$\max_{\vec{P}} \min_{\vec{Q}} \sum_{j=1}^{n} Q_j \sum_{i=1}^{m} A_{ij} P_i = \max_{\vec{P}} \min_{j=1..n} \sum_{i=1}^{m} A_{ij} P_i$$

and thus player 1's goal is to solve

$$\max_{\vec{P}} \min_{j=1..n} E_j(P) \quad \text{where} \quad E_j(\vec{P}) = \sum_{i=1}^{m} A_{ij} P_i$$

and player 2's goal is to solve

$$\min_{\vec{Q}} \max_{i=1..m} F_i(\vec{Q}) \quad \text{where} \quad F_i(\vec{Q}) = \sum_{j=1}^{n} A_{ij} Q_j$$

<u>proof of lemma:</u> We want to show that if $\sum_{i=1}^{m} P_i = 1$ and $\vec{P} \geq 0$ then

$$\max_{\vec{P}} \sum_{i=1}^{m} \phi_i P_i = \max_i \phi_i$$

First, $\exists h$ so that $\max_i \phi_i = \phi_h$. $\quad (\phi_i \leq \phi_h \ \forall i = 1..m)$

$$\Rightarrow \phi_i P_i \leq \phi_h P_i \qquad \text{since } P_i \geq 0 \quad i = 1 \dots m$$

$$\Rightarrow \sum_{i=1}^{m} \phi_i P_i \leq \sum_{i=1}^{m} \phi_h P_i = \phi_h \sum_{i=1}^{m} P_i = \phi_h \quad (\text{since } \textstyle\sum P_i = 1)$$

$$\Rightarrow \text{whatever } \vec{P} \text{ is}, \quad \sum_{i=1}^{m} \phi_i P_i \leq \phi_h = \max_i \phi_i.$$

$$\Rightarrow \max_{\vec{P}} \sum_{i=1}^{m} \phi_i P_i \leq \max_i \phi_i \qquad (*)$$

Now for the other direction. We'll choose a specific mixed strategy,

$$\tilde{p}_i = \begin{cases} 1 & \text{if } i = k \text{ the index where } \\ 0 & \text{if } i \neq k \quad \phi_i \text{ maximized.} \end{cases}$$

$\Rightarrow \sum_{i=1}^{m} \phi_i \tilde{p}_i = \phi_k$. Since this is a specific choice of strategy, we know

$$(**) \quad \max_{\vec{p}} \sum_{i=1}^{m} \phi_i p_i \geq \sum_{i=1}^{m} \phi_i \tilde{p}_i = \phi_k = \max_i \phi_k .$$

Combining $(*)$ and $(**)$ we have

$$\max_{\vec{p}} \sum_{i=1}^{m} \phi_i p_i = \max_i \phi_k$$

The argument for

$$\min_{\vec{Q}} \sum_{j=1}^{n} \psi_j q_j = \min_j \psi_j \quad \text{is nearly identical.}$$

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So Player 1's goal is:
find $\vec{p} \in \mathbb{R}^m$ and $u \in \mathbb{R}$ such that

$$\vec{p} \geq 0, \quad \sum_{i=1}^{m} p_i = 1, \quad u = \max_{\vec{p}} \min_{j=1\ldots n} \sum_{i=1}^{m} A_{ij} p_i$$

How can you write this as a linear programming problem? Player 1 is trying to solve:

maximize u
subject to

$$u \leq \sum_{i=1}^{m} A_{i1} p_i \qquad (j=1)$$

$$u \leq \sum_{i=1}^{m} A_{i2} p_i \qquad (j=2)$$

$$\vdots$$

$$u \leq \sum_{i=1}^{m} A_{in} p_i \qquad (j=n)$$

$$p_1 + p_2 + \ldots + p_m = 1$$

$p_i \geq 0$ for $i=1\ldots m$, u unconstrained.

Denote $(u_*, \vec{p}_*)$ the optimal solution(s) of this linear programming problem. Then
u_* is the expected payoff to player 1
if Player 1 plays the optimal strategy p_*

Similarly, player 2's goal is:

find $\vec{Q} \in \mathbb{R}^n$ and $v \in \mathbb{R}$ such that

$$\vec{Q} \geq 0, \quad \sum_{j=1}^{n} Q_j = 1 \quad \text{and} \quad v = \min_{\vec{Q}} \max_{i=1..m} \sum_{j=1}^{n} A_{ij} Q_j$$

How can you write this as a linear programming problem? Player 2 is trying to solve:

minimize v

subject to

$$\sum_{j=1}^{n} A_{1j} Q_j \leq v \qquad (i = 1)$$

$$\sum_{j=1}^{n} A_{2j} Q_j \leq v \qquad (i = 2)$$

$$\vdots$$

$$\sum_{j=1}^{n} A_{mj} Q_j \leq v \qquad (i = m)$$

$$Q_1 + Q_2 + \dots + Q_n = 1$$

$Q_i \geq 0$ for $i = 1 \dots n$, v unconstrained.

Denote $(v_*, \vec{Q}_*)$ the optimal solution(s) of this linear programming problem. Then v_* is the expected payoff to player 1 if Player 2 plays the optimal strategy $\vec{Q}_*$.

Let's return to our Dr Odd, Dr Even game

$$\begin{array}{cc|ccc|}
 & & \multicolumn{3}{c}{\text{Dr Odd}} \\
 & & 1 & 2 & 3 \\
\hline
 & 1 & 2 & -3 & 2 \\
\text{Dr Even} & 2 & -3 & 4 & -3 \\
 & 3 & 2 & -3 & 6 \\
\hline
\end{array}$$

primal problem

maximize u

such that

$$p_1 + p_2 + p_3 = 1$$

$$u \le +2p_1 - 3p_2 + 2p_3$$

$$u \le -3p_1 + 4p_2 - 3p_3$$

$$u \le 2p_1 - 3p_2 + 6p_3$$

$p_1, p_2, p_3 \ge 0$ u unconstrained.

opt soln

$p_1 = 7/12$

$p_2 = 5/12$

$p_3 = 0$

$u = -1/12$

dual problem

minimize v

such that

$$q_1 + q_2 + q_3 = 1$$

$$+2q_1 - 3q_2 + 2q_3 \le v$$

$$-3q_1 + 4q_2 - 3q_3 \le v$$

$$+2q_1 - 3q_2 + 6q_3 \le v$$

$q_1, q_2, q_3 \ge 0$ v unconstrained

opt soln:

$q_1 = 7/12$

$q_2 = 5/12$

$q_3 = 0$

$v = -1/12$

$\Rightarrow$ the best that Dr Even can hope for is $-1/12$ $\Rightarrow$ Dr Odd will win this game on average. (play many many games)

Question: Are there any matrix games where there is no optimal strategy?

Answer: No!

Theorem: (the von Neumann Minimax Theorem)

Let A be an $m \times n$ matrix game. Then there exist optimal mixed strategies P^* and Q^* for the row player and column player respectively. Furthermore

$$u^* = \max_{\vec{P}} \min_{j=1,..,n} \sum_{i=1}^{m} A_{ij} P_i = \min_{\vec{Q}} \max_{i=1,..,m} \sum_{j=1}^{n} A_{ij} Q_j = v^*$$

where (u^*, P^*) is an optimal solution of the linear programming problem Player 1 has to solve and (v^*, Q^*) is an optimal solution of the linear programming problem Player 2 has to solve. This common value $(u^* = v^*)$ is called the von Neumann value of the game

For example consider payoff matrix

$$\begin{array}{cc} & \text{II} \\ \text{I} & \begin{pmatrix} 2 & 1 & 4 & 2 \\ 1 & 2 & 1 & 1 \\ -2 & 6 & 3 & -2 \\ 3 & -3 & 5 & 1 \\ 1 & 2 & 2 & 1 \end{pmatrix} \end{array}$$

First look for dominating rows.
Is one row larger entry by entry than others?
row 5 dominates row 2 → remove row 2

$$\begin{array}{cc} & \text{II} \\ \text{I} & \begin{pmatrix} 2 & 1 & 4 & 2 \\ -2 & 6 & 3 & -2 \\ 3 & -3 & 5 & 1 \\ 1 & 2 & 2 & 1 \end{pmatrix} \end{array}$$

Now look for a dominating column. Is one column smaller than others?
column 4 is smaller than column 1
→ delete column 1

$$\begin{array}{cc} & \text{II} \\ \text{I} & \begin{pmatrix} 1 & 4 & 2 \\ 6 & 3 & -2 \\ -3 & 5 & 1 \\ 2 & 2 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{c|c}
 & \begin{matrix} q_2 & q_3 & q_4 \end{matrix} \\
\begin{matrix} p_1 \\ p_3 \\ p_4 \\ p_5 \end{matrix} & \begin{pmatrix} 1 & 4 & 2 \\ 6 & 3 & -2 \\ -3 & 5 & 1 \\ 2 & 2 & 1 \end{pmatrix}
\end{array}$$

$$E_2(\vec{p}) = p_1 + 6p_3 - 3p_4 + 2p_5$$
$$E_3(\vec{p}) = 4p_1 + 3p_3 + 5p_4 + 2p_5$$
$$E_4(\vec{p}) = 2p_1 - 2p_3 + p_4 + p_5$$

primal problem:

maximize u

such that

$$p_1 + p_3 + p_4 + p_5 = 1$$
$$u \le E_2(\vec{p})$$
$$u \le E_3(\vec{p})$$
$$u \le E_4(\vec{p})$$

$\vec{p} \ge 0$ u, unconstr

opt soln: $p_1 = 8/9$, $p_3 = 1/9$, $p_4 = 0$, $p_5 = 0$

opt value $1\,5/9$

And recalling that row 2 was removed in going from the original payoff matrix to the reduced payoff matrix, for the original problem $p_2 = 0$ and the optimal solution is

$$\vec{p} = (8/9, 0, 1/9, 0, 0)$$

Note: If you hadn't removed dominated rows and dominating columns, you can still write and solve the linear programming problem. It'll just be bigger and messier, is all.

Dual problem

$$F_1(\vec{Q}) = q_2 + 4q_3 + 2q_4$$
$$F_3(\vec{Q}) = 6q_2 + 3q_3 - 2q_4$$
$$F_4(\vec{Q}) = -3q_2 + 5q_3 + q_4$$
$$F_5(\vec{Q}) = 2q_2 + 2q_3 - q_4$$

minimize v

subject to

$F_1(\vec{Q}) \le v$

$F_3(\vec{Q}) \le v$

$F_4(\vec{Q}) \le v$

$F_5(\vec{Q}) \le v$

$q_2 + q_3 + q_4 = 1$

$q_2, q_3, q_4 \ge 0$ v unconstr.

opt soln

$q_2 = 4/9$

$q_3 = 0$

$q_4 = 5/9$

opt value: $1\,5/9$

(for the full problem, $q_1 = 0$)

So this game will favor player 1.

Other games? Civil War w/ cards.

two players

Katia & Vinh. 52 card deck of cards w/ 26 black cards (spades & clubs) and 26 red cards (hearts & diamonds)

the black suits are ranked higher than the red suits (independent of the denominations, 2 of clubs beats king of hearts.)

Each player antes $\$x$ $(x \ge 0)$ and is then dealt a single card face down.

each player looks at their card

Player 1 now has two options.

Pass : both cards revealed, high hand wins the pot. If hands equal, pot divided evenly

Bet : player adds \$y $(y \geq 0)$

If player I bets, player II has two options

fold : player II loses his ante to player I

see : player II adds \$y to the pot.

If player II sees, both cards are revealed and the high hand wins the pot. (if hands are equal, pot divided equally).

what are the choices for player I ?

PP ← transl : if red then pass, if black, pass

PB ← if red, pass, if black, bet

BP ← if red bet, if black, pass

BB ← if red bet, if black bet,

Player II has four options

FF

FS

SF

SS

Now to compute the payoff matrix. I'll show how to compute the (BP, FS) entry. You can do the others.

player I plays BP, player II plays FS

	I II	I II	I II	I II
hand dealt:	b r	b b	r b	r r
probability:	$\frac{26}{52}\cdot\frac{26}{51}$	$\frac{26}{52}\cdot\frac{25}{51}$	$\frac{26}{52}\cdot\frac{26}{51}$	$\frac{26}{52}\cdot\frac{25}{51}$
action:	P F	P S	B S	B F
outcome for player I:	$+x$	0	$-x-y$	$+x$

Expected winnings for player I:

$$\frac{26}{52}\cdot\frac{26}{51}\cdot x + \frac{26}{52}\cdot\frac{25}{51}\cdot 0 + \frac{26}{52}\cdot\frac{26}{51}(-x-y) + \frac{26}{52}\cdot\frac{25}{51}\cdot x$$

$$= \frac{25x-26y}{102}$$

Continuing in this way, the payoff matrix to player I is

	FF	FS	SF	SS
PP	0	0	0	0
PB	$25x/102$	0	$(25x+26y)/102$	$13y/51$
BP	$77x/102$	$(25x-26y)/102$	$26x/51$	$-13y/51$
BB	x	$(25x-26y)/102$	$(77x+26y)/102$	0

remove row 1 since row 2 is better for player I than row 1.

remove row 3 since row 4 is better for player I than row 3

	FF	FS	SF	SS
PB	$25x/102$	0	$(25x+26y)/102$	$13y/51$
BB	x	$(25x-26y)/102$	$(77x+26y)/102$	0

Remove column 3 since player II is better off playing SS than SS --- player II will pay less \$ to player I

Remove column 1 since player II is better off playing FS than FF --- player II will pay less \$ to player I

	FS	SS
PB	0	$13y/51$
BB	$\frac{25x-26y}{102}$	0

Case 1: $y \geq \frac{25}{26}x$

In this case, the (BB, FS) entry is ≤ 0. And player I will always be better off playing PB. Since player I will always play PB, player II will always play FS and the expected payoff is $0 \Rightarrow$ the game is fair

Case 2: $y < \frac{25}{26} x$

Apply the simplex method and find:

optimal strategy for player I: never play PP or BP

play PB w/ prob. $(25x - 26y)/25x$

play BB w/ prob: $\frac{26}{25} y/x$

expected winnings for player I: $\dfrac{13y(25x-26y)}{1275x}$

optimal strat. for player II: never play FF or SF

play FS w/ prob: $\frac{26}{25} y/x$

play SS w/ prob: $(25x - 26y)/25x$

expected pay to player I: $\dfrac{13y(25x-26y)}{1275x}$

Since $y < \frac{25}{26} x$, we see the expected payoff to player I is > 0. $\Rightarrow$ game biased towards player I if betting doesn't cost too much.

opt play for I: Always bet if black card, bet w/ prob. $\frac{26}{25} y/x$ if red

opt play for II: Always see if black card, fold w/ prob $\frac{26}{25} y/x$ if red.

Example: Suppose fighter planes can be equipped with one of the four armaments: guns, rockets, toss-bombs, and ramming. Then fighters are to be used against bombers of the following three types: full firepower & low speed, partial firepower & medium speed, and no firepower & high speed. We wish to determine the best type of armament for the fighter and the best type of bomber. Assume we are able to establish the effectiveness of each type of fighter versus each type of bomber. This represents the payoff and is measured by a bomber attrition factor per fighter:

		bomber command		
		full fire, low speed	partial fire, med. speed	no fire, high speed
fighter command	1. guns	.30	.25	.15
	2. rockets	.18	.14	.16
	3. toss-bombs	.35	.22	.17
	4. ramming	.21	.16	.10

Find the optimal strategies for the fighter command and the bomber command.

first, we remove dominating columns and dominated rows:

col 3 is better than col 1 for bomber command,

$\Rightarrow$

	q_2	q_3
p_1	.25	.15
p_2	.14	.16
p_3	.22	.17
p_4	.16	.10

row 3 is better than row 2 and row 4 for fighter command

	q_2	q_3
p_1	.25	.15
p_3	.22	.17

"col 3" is better than "col 2" for bomber command

	q_3
p_1	.15
p_3	.17

"row 3" better than "row 1" for fighter command

q_3

$\rightarrow$

p_3 .17

$\Rightarrow$ the optimal strategies $\vec{P}$ and $\vec{Q}$ are pure:

$p_1 = p_2 = p_4 = 0 \qquad p_3 = 1$

$q_1 = q_2 = 0 \qquad q_3 = 1$

(How lucky for us! Was it a miracle? No... this is a case where the game has a "saddle point". In this case you don't need a mixed strategy.)

A guessing game.

Blue secretly picks one of three numbers: 1, 2, 3. Red then proceeds to guess the number picked by announcing his guess. Each time Red announces his guess, Blue answers "high", "low", or "correct", as the case may be. The game continues until Red has guessed correctly. The payoff to Blue is the number of guesses required by Red to identify the number.

A strategy for Blue is the choice of a number 1, 2, or 3.

A strategy for Red may be represented by a triplet $(G; H, L)$ where G is the number guessed the first round, H is the number guessed the second round if Red hears "high" and L is the number guessed if Red hears "low". The game can be terminated in two rounds. Red has 5 strategies

$(1; 0, 2)$

$(1; 0, 3)$

$(2; 1, 3)$

$(3; 1, 0)$

$(3; 2, 0)$

where 0 means "not applicable"

Guessing game payoff.

Red

		(1; 0, 2)	(1; 0, 3)	(2; 1, 3)	(3; 1, 0)	(3; 2, 0)
	1	1	1	2	2	3
Blue	2	2	3	1	3	2
	3	3	2	2	1	1

Remove any dominated rows and dominating columns... there are none!

Blues goal:

max v

$$v \le p_1 + 2p_2 + 3p_3$$
$$v \le p_1 + 3p_2 + 2p_3$$
$$v \le 2p_1 + p_2 + 2p_3$$
$$v \le 2p_1 + 3p_2 + p_3$$
$$v \le 3p_1 + 2p_2 + p_3$$
$$p_1 + p_2 + p_3 = 1$$

$p_1, p_2, p_3 \ge 0$ $\quad v$ unconstr.

opt soln:

$p_1 = 2/5$

$p_2 = 1/5$

$p_3 = 2/5$

obj. value $= 1\ 4/5$

Red's goal

min u

$$q_1 + q_2 + 2q_3 + 2q_4 + 3q_5 \le u$$
$$2q_1 + 3q_2 + q_3 + 3q_4 + 2q_5 \le u$$
$$3q_1 + 2q_2 + 2q_3 + q_4 + q_5 \le u$$
$$q_1 + q_2 + q_3 + q_4 + q_5 = 1$$

$q_1, q_2, q_3, q_4, q_5 \ge 0$ $\quad u$ unconstr.

opt soln:

$q_1 = 0$

$q_2 = 1/5$

$q_3 = 3/5$

$q_4 = 1/5$

$q_5 = 0$

obj. value $= 1\ 4/5$

I've been referring to Row player's problem as "primal" and the column player's problem as "dual". Is this legitimate? Let's see!

If you fix a $\vec{P} \geq 0$ that satisfies

$$\sum_{i=1}^{m} P_i = 1$$

then you want to find the largest value of u that will satisfy

$$u \leq E_1(\vec{P})$$
$$u \leq E_2(\vec{P})$$
$$\vdots$$
$$u \leq E_n(\vec{P})$$

(i.e $u = \min_{j=1..n} E_j(\vec{P})$).

then you want to let $\vec{P}$ vary so you can find the largest possible u.

i.e

maximize u
such that

$$\sum_{i=1}^{m} P_i = 1$$
$$u \leq E_1(\vec{P})$$
$$u \leq E_2(\vec{P})$$
$$\vdots$$
$$u \leq E_n(\vec{P})$$

$\vec{P} \geq 0$, u unconstrained.

this is the linear programming problem the row player must solve

How do we write this in terms of the matrix A?

$$E_j(\vec{P}) = \sum_{i=1}^{m} A_{ij} P_i$$
$$= \sum_{i=1}^{m} (A^T)_{ji} P_i$$
$$= (A^T \vec{P})_j$$

so the constraint

$$u \leq E_j(\vec{P})$$

can be written as

$$0 \leq (A^T \vec{P})_j - u \qquad j = 1 .. n$$

So the row player's problem is

maximize u $\longrightarrow$ (minimize $-u$)

such that

$$\begin{pmatrix} & & & & -1 \\ & A^T & & & -1 \\ & & & & -1 \\ & & & & \vdots \\ -1 & -1 & -1 & \cdots -1 & 0 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_m \\ u \end{pmatrix} \begin{matrix} \geq 0 \\ \geq 0 \\ \geq 0 \\ \geq 0 \\ = -1 \end{matrix}$$

$P_1 \ldots P_m \geq 0$ $\quad u$ unconstrained.

The dual problem is

maximize $-v$

subject to

$$\left(\begin{array}{ccc|c} & & & -1 \\ & A & & -1 \\ & & & -1 \\ & & & -1 \\ \hline -1 & -1 \cdots & -1 & 0 \end{array}\right) \begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \\ v \end{pmatrix} \begin{matrix} \leq \\ \leq \\ \leq \\ \vdots \\ = \end{matrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}$$

$Q_i \geq 0$, v unconstrained.

i.e.

the dual problem is

minimize v subject to

$$\sum_{j=1}^{n} Q_j = 1$$

$$F_i(\vec{Q}) - v \leq 0 \qquad i = 1..m$$

$\vec{Q} \geq 0$, Q unconstrained

What does this mean? fix $\vec{Q}$. then we want the smallest v such that

$$F_i(\vec{Q}) \leq v \qquad i = 1 \ldots m$$

i.e. $v = \max\limits_{i=1..m} F_i(\vec{Q})$. Now let $\vec{Q}$ vary.

we get optimal solution $\min\limits_{\vec{Q}} \max\limits_{i=1..m} F_i(\vec{Q})$.

We see that the dual of the linear programming problem for the row player is the linear programming problem for the column player.

$\Rightarrow$ Duality theorem tells us that if the row player plays the optimal strategy

$$(u^*, P^*)$$

and the column player plays the optimal strategy

$$(v^*, Q^*)$$

then the obj. function value of primal problem = obj. function value of dual problem.

That is, $u^* = v^*$