APM 236 Second Midterm 3/20/2002 points possible

You may not use calculators, cell phones, or PDAs during the exam. Partial credit is possible. Please read the entire test over before starting. Please put a box around your solutions so that the grader can find them easily.

Print your name clearly:

Print your student number clearly:

Please sign here:

Problem 1

Problem 2

Problem 3

Problem 4

Problem 5

Problem 6

Total

1. (5 points) Consider the following linear programming problem:

$$\text{Maximize } 10x_1 + 3x_2$$
$$\text{Subject to}$$

$$\begin{array}{rrrcr} x_1 & +2x_2 & +3x_3 & \leq & 1 \\ & 4x_2 & & \leq & 2 \\ -x_1 & & +9x_3 & \leq & 0 \\ x_1 & -x_2 & & \leq & -4 \end{array}$$

where $x_1, x_2, x_3 \geq 0$.

What is the dual problem?

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ -1 & 0 & 9 \\ 1 & -1 & 0 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -4 \end{pmatrix} \quad \vec{c} = \begin{pmatrix} 10 \\ 3 \\ 0 \end{pmatrix}$$

dual problem:

minimize $w_1 + 2w_2 - 4w_4$

subject to

$$\begin{array}{rrrrcr} w_1 & & -w_3 & +w_4 & \geq & 10 \\ 2w_1 & +4w_2 & & -w_4 & \geq & 3 \\ 3w_1 & & +9w_3 & & \geq & 0 \end{array}$$

$w_1, w_2, w_3, w_4 \geq 0$

2. (10 points) Consider the following linear programming problem:

$$\text{Maximize } x_2 - x_3 + 3x_4$$
$$\text{Subject to}$$
$$\begin{array}{rrrrcr} 3x_1 & -x_2 & & +2x_4 & \leq & -10 \\ 6x_1 & +x_2 & +x_3 & +2x_4 & \leq & 0 \end{array}$$
$$\text{where } x_1, x_2, x_3, x_4 \geq 0.$$

You want to use the simplex method to find an optimal solution. Give the first two tableaux.

First make RHS of constraint ≥ 0

$$-3x_1 + x_2 \quad -2x_4 \geq 10$$
$$6x_1 + x_2 + x_3 + 2x_4 \leq 0$$

Introduce slack variables:

$$-3x_1 + x_2 \quad -2x_4 - x_5 = 10$$
$$6x_1 + x_2 + x_3 + 2x_4 \quad + x_6 = 0$$

Introduce artificial variable

$$-3x_1 + x_2 \quad -2x_4 - x_5 \quad + y = 10$$
$$6x_1 + x_2 + x_3 + 2x_4 \quad + x_6 \quad = 0$$

We start phase 1. We want to maximize the objective function $z' = -y$. But it should be in terms of the nonbasic variables only. Use eqn 1: $-3x_1 + x_2 - 2x_4 - x_5 - 10 = -y$

more room to work on back!

Initial tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	y	
y	-3	1	0	-2	-1	0	1	10
x_3	6	1	1	2	0	1	0	0
	3	-1	0	2	1	0	0	-10

note! could take x_6 as basic variable here!

take x_2 as incoming variable and x_3 (or x_6) as outgoing variable.

	x_1	x_2	x_3	x_4	x_5	x_6	y	
y	-9	1	-1	-4	-1	-1	1	10
x_2	6	0	1	2	0	1	0	0
	9	0	1	4	1	1	0	-10

Phase 1 terminated but not successfully

$\therefore$ there are no feasible solutions to the original problem. ☹

3. (10 points) Consider the following linear programming problem:

$$\text{Maximize } x_1 + x_2$$
$$\text{Subject to}$$
$$\begin{array}{rrcl} -x_1 & +x_2 & \leq & 1 \\ 4x_1 & -x_2 & \leq & 8 \end{array}$$
$$\text{where } x_1, x_2 \geq 0.$$

At one point during the simplex method, the basic variables are x_2 and x_4. Give the tableau at that time.

The RHS is already ≥0. So we start by adding slack variables

$$-x_1 + x_2 + x_3 \quad = 1$$
$$4x_1 - x_2 \quad + x_4 = 8$$

so $\vec{b} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$ $A = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 4 & -1 & 0 & 1 \end{pmatrix}$ $\vec{c} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

the basic variables are x_2 and x_4 so

$B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

we need *more room to work on back!*

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix}$$

So the desired tableau is

		1	1	0	0	
		x_1	x_2	x_3	x_4	
1	x_2	-1	1	1	0	1
0	x_4	3	0	1	1	9
		-2	0	1	0	1

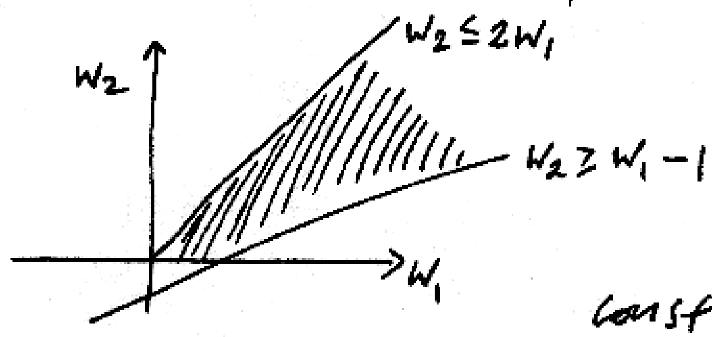
$$(obj)_1 = 1\cdot(-1) + 0\cdot 3 - 1 = -2$$

$$(obj)_3 = 1\cdot 1 + 0\cdot 1 - 0 = 1$$

$$(obj)_5 = 1\cdot 1 + 0\cdot 9 = 1$$

4. (10 points) Give an example of a linear programming problem whose dual has feasible solutions but no finite optimal solution.

I'll do this problem by finding an unbounded problem and then finding what it's the dual of.



constraints

$w_2 \le 2w_1$

$w_2 \ge w_1 - 1$

become

$0 \le 2w_1 - w_2$

$-1 \le -w_1 + w_2$

$\Rightarrow A^T\vec{w} \ge \vec{c}$ where

$A^T = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ $\vec{c} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

I want to minimize $-w_2$ since this will have no finite optimal solution.

$\Rightarrow \vec{b}^T\vec{w} = -w_2$

and $\vec{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

take the primal problem

maximize $-x_2$

subject to $2x_1 - x_2 \le 0$

$-x_1 + x_2 \le -1$

$x_1, x_2 \ge 0.$

this is maximize $\vec{c}^T\vec{x}$ subject to $A\vec{x} \le \vec{b}$, $\vec{x} \ge 0$

5. (5 points) Complete the following tableau:

		1	?	1	?	
c_B		x_1	x_2	x_3	x_4	x_B
?	?	1	0	$-3/4$	0	$5/2$
2	?	0	0	?	1	$15/2$
0	?	0	1	0	0	0
		?	?	$3/4$	?	?

		1	0	1	2	
		x_1	x_2	x_3	x_4	
1	x_1	1	0	$-3/4$	0	$5/2$
2	x_4	0	0	$5/4$	1	$15/2$
0	x_2	0	1	0	0	0
		1	0	$3/4$	0	$35/2$

$$1 \cdot \frac{5}{2} + 2 \cdot \frac{15}{2} = 35/2$$

$$-\frac{3}{4} + 2 \cdot ? - 1 = 3/4$$

$$2 \cdot ? - 7/4 = 3/4$$

$$2 \cdot ? = 10/4$$

$$? = 5/4$$

6. Consider the following tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
x_3	0	0	1	0	0	-2	0	2	0
x_5	0	-2	0	10	1	2	0	0	1
x_7	0	-1	0	10	0	4	1	-5	1
x_1	1	2	0	-3	0	-4	0	1	0
	0	-4	0	-2	0	-1	0	-10	10

a. (1 pt) There are four possible choices of incoming variables. What are they?

x_2, x_4, x_6, x_8

b. (6 pt) For each possible incoming variable, give the outgoing variable(s) that are possible. (Think something like, "If x_1 is incoming then I have to look at x_3, x_2, x_5, and x_8. I know that x_3 and x_2 are not going to be outgoing because... And x_5 won't be outgoing because... So for x_1 the only possible outgoing variable is...")

x_2 incoming $\Rightarrow$ x_1 outgoing

x_4 incoming $\Rightarrow$ either x_5 or x_7 outgoing

x_6 incoming $\Rightarrow$ x_7 outgoing

x_8 incoming $\Rightarrow$ either x_3 or x_1 outgoing

c. (3 pt) You now have a collection of pairs of incoming and outgoing variables. List them. Now, discuss which pair you would choose and why you'd choose it.

(incoming, outgoing)

$(x_2, x_1) \rightarrow$ would lead to no incr of obj. value

$(x_4, x_5) \rightarrow$ } both incr. obj. value by $1/5$

$(x_4, x_7) \rightarrow$

$(x_6, x_7) \rightarrow$ increase obj. value by $1/4$

$(x_8, x_3) \rightarrow$ } would lead to no increase of obj. value.

$(x_8, x_1) \rightarrow$

I'd take x_6 as the incoming variable and x_7 as the outgoing variable. this choice would increase the objective function value fastest at this step.