

**You may not use calculators, cell phones, or PDAs during the exam. Partial credit is possible. Please read the entire test over before starting. Please put a box around your solutions so that the grader can find them easily.**

Print your name clearly:

Print your student number clearly:

Please sign here:

Problem 1

Problem 2

Problem 3

Problem 4

Problem 5

Problem 6

Total

1. (5 points) Consider the following linear programming problem:

Maximize  $4x_2 + x_3$

Subject to

$$\begin{array}{rcll} x_1 & +x_2 & +x_3 & \leq & 1 \\ x_1 & & & \leq & 9 \\ & -x_2 & +3x_3 & \leq & -2 \\ x_1 & & -x_3 & \leq & -2 \end{array}$$

where  $x_1, x_2, x_3 \geq 0$ .

What is the dual problem?

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 1 \\ 9 \\ -2 \\ -2 \end{pmatrix} \quad \vec{c} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$$

dual problem:

minimize  $w_1 + 9w_2 - 2w_3 - 2w_4$

subject to

$$w_1 + w_2 + w_4 \geq 0$$

$$w_1 - w_3 \geq 4$$

$$w_1 + 3w_3 - w_4 \geq 1$$

$$w_1, w_2, w_3, w_4 \geq 0$$

2. (10 points) Consider the following linear programming problem:

$$\begin{aligned} & \text{Maximize } x_2 - x_3 + 3x_4 \\ & \text{Subject to} \\ & -x_1 \quad -x_3 + x_4 \geq 4 \\ & x_1 - x_2 + 2x_3 + x_4 \geq -2 \\ & \text{where } x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

You want to use the simplex method to find an optimal solution. Give the first two tableaux.

First, make RHS of constraint  $\geq 0$

$$\begin{aligned} -x_1 \quad -x_3 + x_4 & \geq 4 \\ -x_1 + x_2 - 2x_3 - x_4 & \leq 2 \end{aligned}$$

Introduce slack variables:

$$\begin{aligned} -x_1 \quad -x_3 + x_4 - x_5 & = 4 \\ -x_1 + x_2 - 2x_3 - x_4 + x_6 & = 2 \end{aligned}$$

Introduce artificial variable:

$$\begin{aligned} -x_1 \quad -x_3 + x_4 - x_5 + y & = 4 \\ -x_1 + x_2 - 2x_3 - x_4 + x_6 & = 2 \end{aligned}$$

We start phase 1. We want to maximize the objective function  $z' = -y$ . But it should be in terms of the nonbasic variables only.

Use eqn 1:  $-x_1 - x_3 + x_4 - x_5 - 4 = -y$

more room to work on back!

Initial tableau

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$y$	
$y$	-1	0	-1	1	-1	0	1	4
$x_2$	-1	1	-2	-1	0	1	0	2
	1	0	1	-1	1	0	0	-4

note! could take  $x_6$  as basic variable here.

Take  $x_4$  as incoming variable and  $y$  as outgoing variable.

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$y$	
$x_4$	-1	0	-1	1	-1	0	1	4
$x_2$	-2	1	-3	0	-1	1	1	6
	0	0	0	0	0	0	1	0

if you'd taken the initial basic variables as  $y$  and  $x_6$  then the second tableau would be

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$y$	
$x_4$	-1	0	-1	1	-1	0	1	4
$x_6$	-2	1	-3	0	-1	1	1	6
	0	0	0	0	0	0	1	0

phase 1 terminated successfully and we can continue to phase 2. 😊

3. (10 points) Consider the following linear programming problem:

Maximize  $x_1 - x_2$

Subject to

$$-x_1 + x_2 \leq 1$$

$$2x_1 + x_2 \leq 16$$

where  $x_1, x_2 \geq 0$ .

At one point during the simplex method, the basic variables are  $x_3$  and  $x_1$ . Give the tableau at that time.

The RHS is already  $\geq 0$ . So we start by adding slack variables

$$\begin{aligned} -x_1 + x_2 + x_3 &= 1 \\ 2x_1 + x_2 + x_4 &= 16 \end{aligned}$$

$$\text{so } \vec{b} = \begin{pmatrix} 1 \\ 16 \end{pmatrix} \quad A = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \quad \vec{c} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

If basic variables are  $x_3$  and  $x_1$ , then

$$B = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad B^{-1} = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}$$

$$\text{we need } \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$$

more room to work on back!

$$\begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \quad \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 16 \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \end{pmatrix}$$

So the desired tableau is:

		1	-1	0	0	
		$x_1$	$x_2$	$x_3$	$x_4$	
0	$x_3$	0	$\frac{3}{2}$	1	$\frac{1}{2}$	9
1	$x_1$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	8
		0	$\frac{3}{2}$	0	$\frac{1}{2}$	8

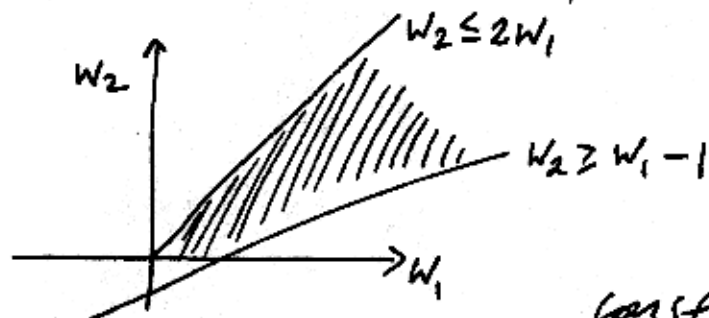
$$(obj)_2 = 0 \cdot \frac{3}{2} + 1 \cdot \frac{1}{2} - (-1) = \frac{3}{2}$$

$$(obj)_1 = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} - 0 = \frac{1}{2}$$

$$(obj)_S = 0 \cdot 9 + 8 \cdot 1 - 0 = 8$$

4. (10 points) Give an example of a linear programming problem whose dual has feasible solutions but no finite optimal solution.

I'll do this problem by finding an unbounded problem and then finding what it's the dual of.



constraints

$$w_2 \leq 2w_1$$

$$w_2 \geq w_1 - 1$$

I want to minimize  $-w_2$  since this will have no finite optimal solution.

become

$$0 \leq 2w_1 - w_2$$

$$-1 \leq -w_1 + w_2$$

$$\Rightarrow \vec{A}^T \vec{w} \geq \vec{c} \text{ where}$$

$$\Rightarrow \vec{b}^T \vec{w} = -w_2$$

and  $\vec{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$$A^T = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \quad \vec{c} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

take the primal problem

$$\begin{aligned} &\text{maximize } -x_2 \\ &\text{subject to } 2x_1 - x_2 \leq 0 \\ &\quad \quad \quad -x_1 + x_2 \leq -1 \\ &\quad \quad \quad x_1, x_2 \geq 0. \end{aligned}$$

} this is maximize  $\vec{c}^T \vec{x}$   
subject to  $A\vec{x} \leq \vec{b}$   
 $\vec{x} \geq 0$

