APM 236 Second Midterm 3/20/2002 points possible

You may not use calculators, cell phones, or PDAs during the exam. Partial credit is possible. Please read the entire test over before starting. Please put a box around your solutions so that the grader can find them easily.

Print your name clearly:

Print your student number clearly:

Please sign here:

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	
Total	

1. (5 points) Consider the following linear programming problem:

$$\text{Maximize } 4x_2 + x_3$$
$$\text{Subject to}$$

$$\begin{array}{rrrcr} x_1 & +x_2 & +x_3 & \leq & 1 \\ x_1 & & & \leq & 9 \\ & -x_2 & +3x_3 & \leq & -2 \\ x_1 & & -x_3 & \leq & -2 \end{array}$$

$$\text{where } x_1, x_2, x_3 \geq 0.$$

What is the dual problem?

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 1 \\ 9 \\ -2 \\ -2 \end{pmatrix} \quad \vec{c} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$$

dual problem:

minimize $W_1 + 9W_2 - 2W_3 - 2W_4$

subject to

$$\begin{array}{lllll} W_1 + W_2 & & +W_4 & \geq & 0 \\ W_1 & -W_3 & & \geq & 4 \\ & +3W_3 & -W_4 & \geq & 1 \\ W_1 & & & & \end{array}$$

$W_1, W_2, W_3, W_4 \geq 0$

2. (10 points) Consider the following linear programming problem:

$$\text{Maximize } x_2 - x_3 + 3x_4$$
$$\text{Subject to}$$
$$\begin{array}{rrrrcr} -x_1 & & -x_3 & +x_4 & \geq & 4 \\ x_1 & -x_2 & +2x_3 & +x_4 & \geq & -2 \end{array}$$
$$\text{where } x_1, x_2, x_3, x_4 \geq 0.$$

You want to use the simplex method to find an optimal solution. Give the first two tableaux.

First, make RHS of constraint ≥ 0

$$\begin{array}{rrrrcr} -x_1 & & -x_3 & +x_4 & \geq & 4 \\ -x_1 & +x_2 & -2x_3 & -x_4 & \leq & 2 \end{array}$$

Introduce slack variables:

$$\begin{array}{rrrrrrcr} -x_1 & & -x_3 & +x_4 & -x_5 & & = & 4 \\ -x_1 & +x_2 & -2x_3 & -x_4 & & +x_6 & = & 2 \end{array}$$

Introduce artificial variable:

$$\begin{array}{rrrrrrrcr} -x_1 & & -x_3 & +x_4 & -x_5 & & +y & = & 4 \\ -x_1 & +x_2 & -2x_3 & -x_4 & & +x_6 & & = & 2 \end{array}$$

We start phase 1. We want to maximize the objective function $z' = -y$. But it should be in terms of the nonbasic variables only.

Use eqn 1: $-x_1 - x_3 + x_4 - x_5 - 4 = -y$

more room to work on back!

Initial tableau

	x_1	x_2	x_3	x_4	x_5	x_6	y	
y	-1	0	-1	1	-1	0	1	4
x_2	-1	1	-2	-1	0	1	0	2
	1	0	1	-1	1	0	0	-4

note! could take x_6 as basic variable here.

Take x_4 as incoming variable and y as outgoing variable.

	x_1	x_2	x_3	x_4	x_5	x_6	y	
x_4	-1	0	-1	1	-1	0	1	4
x_2	-2	1	-3	0	-1	1	1	6
	0	0	0	0	0	0	1	0

If you'd taken the initial basic variables as y and x_6 then the second tableau would be

	x_1	x_2	x_3	x_4	x_5	x_6	y	
x_4	-1	0	-1	1	-1	0	1	4
x_6	-2	1	-3	0	-1	1	1	6
	0	0	0	0	0	0	1	0

phase 1 terminated successfully and we can continue to phase 2.

3. (10 points) Consider the following linear programming problem:

$$\text{Maximize } x_1 - x_2$$
$$\text{Subject to}$$
$$\begin{array}{rrcr} -x_1 & +x_2 & \leq & 1 \\ 2x_1 & +x_2 & \leq & 16 \end{array}$$
$$\text{where } x_1, x_2 \geq 0.$$

At one point during the simplex method, the basic variables are x_3 and x_1. Give the tableau at that time.

the RHS is already ≥ 0. So we start by adding slack variables

$$-x_1 + x_2 + x_3 \quad = 1$$
$$2x_1 + x_2 \quad + x_4 = 16$$

so $\vec{b} = \begin{pmatrix} 1 \\ 16 \end{pmatrix}$ $A = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$ $\vec{c} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

if basic variables are x_3 and x_1 then

$B = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}$

we need $\begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$

more room to work on back!

$\begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ $\begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 16 \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \end{pmatrix}$

so the desired tableau is:

		1	-1	0	0	
		x_1	x_2	x_3	x_4	
0	x_3	0	3/2	1	1/2	9
1	x_1	1	1/2	0	1/2	8
		0	3/2	0	1/2	8

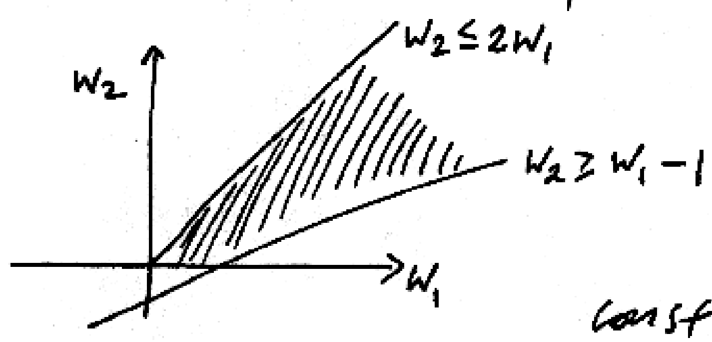
$$(obj)_2 = 0\cdot\frac{3}{2} + 1\cdot\frac{1}{2} - (-1) = 3/2$$

$$(obj)_4 = 0\cdot\frac{1}{2} + 1\cdot\frac{1}{2} - 0 = 1/2$$

$$(obj)_5 = 0\cdot 9 + 8\cdot 1 - 0 = 8$$

4. (10 points) Give an example of a linear programming problem whose dual has feasible solutions but no finite optimal solution.

I'll do this problem by finding an unbounded problem and then finding what it's the dual of.



constraints

$w_2 \le 2w_1$

$w_2 \ge w_1 - 1$

become

$0 \le 2w_1 - w_2$

$-1 \le -w_1 + w_2$

I want to minimize $-w_2$ since this will have no finite optimal solution.

$\Rightarrow \vec{b}^T \vec{w} = -w_2$

and $\vec{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\Rightarrow \vec{A}^T \vec{w} \ge \vec{c}$ where

$A^T = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ $\vec{c} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

take the primal problem

maximize $-x_2$

subject to $2x_1 - x_2 \le 0$

$-x_1 + x_2 \le -1$

$x_1, x_2 \ge 0.$

this is maximize $\vec{c}^T \vec{x}$

subject to

$A\vec{x} \le \vec{b}$

$\vec{x} \ge 0$

5. (5 points) Complete the following tableau:

c_B		1 x_1	2 x_2	? x_3	1 x_4	x_B
1	?	1	3/5	0	0	?
?	?	0	4/5	0	1	6
0	?	0	0	1	0	0
		?	?	?	?	13

		1 X_1	2 X_2	0 X_3	1 X_4	
1	X_1	1	3/5	0	0	7
1	X_4	0	4/5	0	1	6
0	X_3	0	0	1	0	0
		0	$-2/5$	0	0	13

$$1 \cdot \frac{3}{5} + 1 \cdot \frac{4}{5} - 2 = \frac{8}{5} - 2 = -2/5$$

$$1 \cdot ? + 1 \cdot 6 = 13$$

$$\Rightarrow ? = 7$$

6. Consider the following tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
x_3	1	0	1	-2	0	6	0	0	2
x_2	-2	1	0	2	0	-3	-2	0	0
x_5	2	0	0	0	1	6	-3	0	2
x_8	0	0	0	4	0	-1	9	1	0
	-2	0	0	-1	0	-3	-1	0	9

a. (1 pt) There are four possible choices of incoming variables. What are they?

X_1, X_4, X_6, X_7

b. (6 pt) For each possible incoming variable, give the outgoing variable(s) that are possible. (Think something like, "If x_1 is incoming then I have to look at x_3, x_2, x_5, and x_8. I know that x_3 and x_2 are not going to be outgoing because... And x_5 won't be outgoing because... So for x_1 the only possible outgoing variable is...")

X_1 incoming $\Rightarrow$ X_5 outgoing

X_4 incoming $\Rightarrow$ X_2 or X_8 outgoing

X_6 incoming $\Rightarrow$ X_3 or X_5 outgoing

X_7 incoming $\Rightarrow$ X_8 outgoing

c. (3 pt) You now have a collection of pairs of incoming and outgoing variables. List them. Now, discuss which pair you would choose and why you'd choose it.

(incoming, outgoing)

(x_1, x_5) → would increase obj. value by 2

(x_4, x_2) →
(x_4, x_8) → would lead to no increase of objective funct. value

(x_6, x_3) →
(x_6, x_5) → would increase obj. function value by 1

(x_7, x_8) → would lead to no increase of obj. function value.

I'd take x_1 as the incoming variable and x_5 as the outgoing variable. This choice would increase the objective function value fastest at this step