ANSWER KEY

white/green test

1

APM 236 First Midterm (February 6, 2002) 35 points possible

You may not use calculators, cell phones, or PDAs during the exam. Partial credit is possible. Please read the entire test over before starting. Please put a box around your solutions so that the grader can find them easily.

Print your name clearly:

Please sign here:

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1. (3 points) Consider the following linear programming problem:

$$\begin{aligned} &\text{Maximize } x - 9y + 2z \\ &\text{subject to} \\ x + 7y - 2z &\leq 5 \\ 4x - y + z &\geq 6 \\ 2x + y &\leq 0 \\ x, y, z &\geq 0. \end{aligned}$$

Write the problem as a linear programming problem in **canonical** form.

in standard form:

Maximize $x - 9y + 2z$
subject to

$$\begin{aligned} x + 7y - 2z &\leq 5 \\ -4x + y - z &\leq -6 \\ 2x + y &\leq 0 \\ x, y, z &\geq 0 \end{aligned}$$

in canonical form:

Maximize $x - 9y + 2z$
subject to

$$\begin{aligned} x + 7y - 2z + u &= 5 \\ -4x + y - z + v &= -6 \\ 2x + y + w &= 0 \\ x, y, z, u, v, w &\geq 0 \end{aligned}$$

2. (8 points) The Soft Suds Brewing and Bottling Company, because of faulty planning, was not prepared for the Operations Research Department. There was to be a big party at Stanford University[1], and Gus Guzzler, the manager, knew that Soft Suds would be called upon to supply the refreshments. However, the raw materials required had not been ordered and could not be obtained before the party. Gus took an inventory of the available supplies and found the following:

Malt	80 units
Hops	60 units
Yeast	75 units

Soft suds produces two types of pick-me-ups: light beer and dark beer, with the following specifications:

	Requirement per gallon		
	Malt	Hops	Yeast
Light beer	4	2	3
Dark beer	8	12	12

The light beer brings \$3.00/gallon profit, the dark beer \$1.00/gallon profit. Knowing the O.R. department will buy whatever is made available, formulate the linear program Gus must solve to maximize his profits, and solve it graphically. Be sure to define all your variables.

x = gallons light beer
y = gallons dark beer

units of malt used = $4x + 8y \leq 80$
units of Hops used = $2x + 12y \leq 60$
units of yeast used = $3x + 12y \leq 75$

objective function: $3x + 1 \cdot y$

→

[1]This problem is taken from Dantzig's text book. He's the inventor of the simplex method and is an O.R. professor at Stanford.

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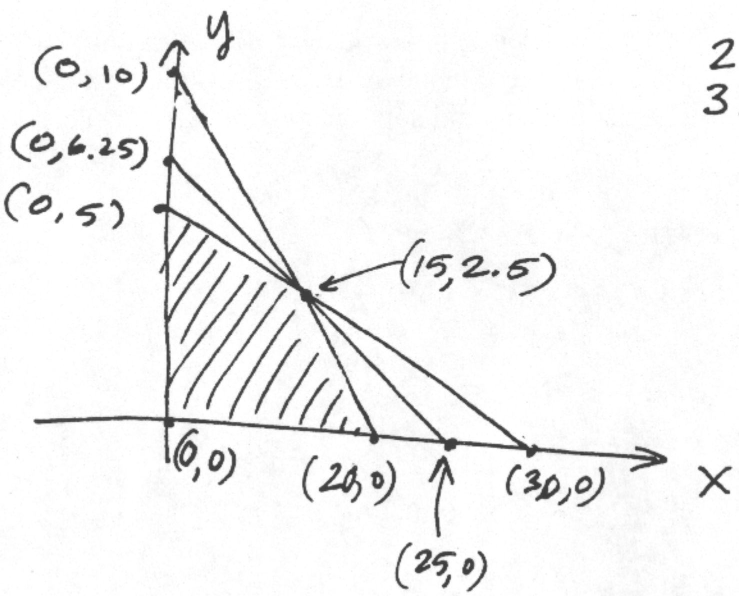
Maximize $3x+y$

subject to

$$4x+8y \le 80$$
$$2x+12y \le 60$$
$$3x+12y \le 75$$



$2x+12y=60$
$3x+12y=75$
$\rightarrow x=15$
$y=2\frac{1}{2}$

4 vertices/extreme points

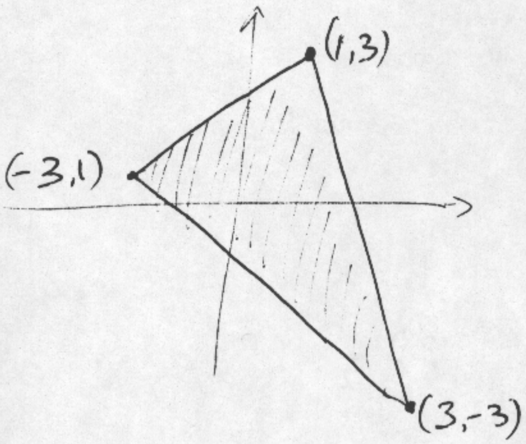
	$\vec{c}^{\,T}\vec{x}$
$(0,0) \rightarrow$	0
$(20,0) \rightarrow$	60
$(15,2\frac{1}{2}) \rightarrow$	$47\frac{1}{2}$
$(0,5) \rightarrow$	5

max value = 60 occurs at (20,0)

3. Consider the convex region shown below.

   a. (3 points) Find a 3×2 matrix A and a vector $\vec{b}$ so that $A\vec{x} \leq \vec{b}$ describes this region. **Note: Choose A and $\vec{b}$ so that all of the entries of $\vec{b}$ are either 1 or -1.**

   b. (3 points) Find a 4×2 matrix A and a vector $\vec{b}$ so that $A\vec{x} \leq \vec{b}$ describes this region. **Note: Choose A and $\vec{b}$ so that all of the entries of $\vec{b}$ are either 1 or -1.**



top line:

$$y = \frac{3-1}{1+3}(x-1)+3$$

right side:

$$y = \frac{-3-3}{3-1}(x-1)+3$$

bottom side:

$$y = \frac{-3-1}{3+3}(x-3)-3$$

$$\begin{cases} y \leq \frac{1}{2}x + 5/2 \\ \cancel{y} \leq -3x+6 \\ y \geq -\frac{2}{3}x - 1 \end{cases} \Rightarrow \begin{cases} -\frac{1}{2}x + y \leq \frac{5}{2} \\ 3x + y \leq 6 \\ -\frac{2}{3}x - y \leq 1 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} -\frac{1}{5}x + \frac{2}{5}y \leq 1 \\ \frac{1}{2}x + \frac{1}{6}y \leq 1 \\ -\frac{2}{3}x - y \leq 1 \end{cases}$$

a: $\begin{pmatrix} -1/5 & \frac{2}{5} \\ 1/2 & 1/6 \\ -2/3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

b: $\begin{pmatrix} -1/5 & 2/5 \\ 1/2 & 1/6 \\ -2/3 & -1 \\ 1/10 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

note: many possible ans. for 4^{th} row!

4. Consider the two LP problems with two unknowns ($\vec{x} = (x_1, x_2)$):

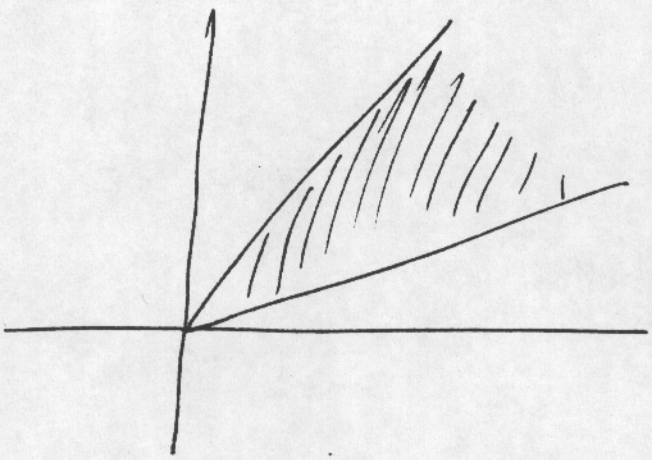
Problem 1:

$$\text{Maximize } \vec{c}^T\vec{x}$$
$$\text{subject to } A\vec{x} \leq \vec{b}$$

Problem 2:

$$\text{Minimize } \vec{c}^T\vec{x}$$
$$\text{subject to } A\vec{x} \leq \vec{b}$$

a. (3 points) Give an example of A, $\vec{b}$, and $\vec{c}$ so neither Problem 1 nor Problem 2 has a solution. Explain your choice graphically.



$$\begin{cases} y \leq x \\ y \geq \frac{x}{2} \end{cases}$$

$\downarrow$

$$\begin{cases} -x + y \leq 0 \\ \frac{x}{2} - y \leq 0 \end{cases}$$

$$A = \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & -1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

note:

many possible answers to this problem!

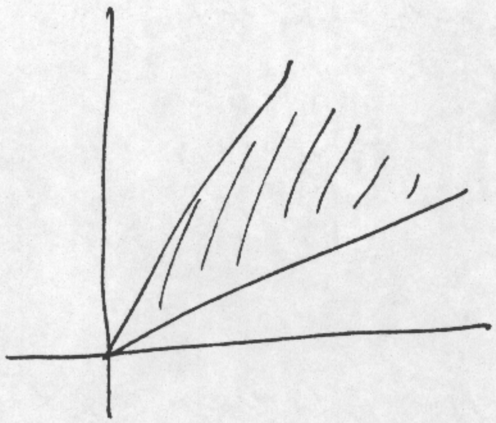
if $\vec{c}^T\vec{x} = x - y$

then if $x \to \infty$ $\vec{c}^T\vec{x} \to \infty$

if $y \to \infty$ $\vec{c}^T\vec{x} \to -\infty$

no max and no min

b. (3 points) Give an example of A, $\vec{b}$, and $\vec{c}$ so that Problem 1 has a solution, but Problem 2 has no solution. Explain your choice graphically.



same A and $\vec{b}$ as before.

take $\vec{c}^T\vec{x} = -x - y$

then as $x \to \infty$ or $y \to \infty$

$\vec{c}^T\vec{x} \to -\infty$.

there's no problem w/ finding the maximum value. It's 0 and occurs at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. There is no minimum reached, however.

note: many possible answers to this problem!

5. (6 points) Find two basic feasible solutions of

$$A\vec{x} = \vec{b}$$

where

$$A = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{matrix} \\ \begin{pmatrix} 0 & 0 & 6 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

and

$$\vec{b}^T = (7, 4, 3).$$

Note: there are $\binom{6}{3} = 20$ different 3×3 matrices to consider.

The four most popular will be

$$\begin{pmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_6 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 \\ -2 \\ 0 \\ 0 \\ 0 \\ 7 \end{pmatrix}$$ not feasible!

$$\begin{pmatrix} 0 & 6 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 \\ 4 \\ -3 \\ 0 \\ 0 \\ 25 \end{pmatrix}$$ not feasible!

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \\ x_6 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 \\ 4 \\ 0 \\ +3 \\ 0 \\ 7 \end{pmatrix}$$ feasible!

$$\begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 \\ 4 \\ 7/6 \\ 25/6 \\ 0 \\ 0 \end{pmatrix}$$ feasible!

6. (6 points) Consider the linear programming problem

$$\text{Maximize } \vec{c}^T\vec{x}$$
$$\text{subject to } A\vec{x} \leq \vec{b}.$$

Show that if there are two distinct points $\vec{x}_1$ and $\vec{x}_2$ where the objective function achieves the maximum value M then there are infinitely points where the value M is achieved. *Note: the points being "distinct" just means $\vec{x}_1 \neq \vec{x}_2$.*

know $\vec{c}^T\vec{x}_1 = M$

$\vec{c}^T\vec{x}_2 = M$

let $\vec{y} = \lambda\vec{x}_1 + (1-\lambda)\vec{x}_2$

then

$$\vec{c}^T\vec{y} = \lambda\vec{c}^T\vec{x}_1 + (1-\lambda)\vec{c}^T\vec{x}_2$$
$$= \lambda M + (1-\lambda)M = M$$

so any point on the line that runs through $\vec{x}_1$ and $\vec{x}_2$ will have $\vec{c}^T\vec{x} = M$. (Not surprising since the level set is a hyperplane.)