answer key
blue/yellow test

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APM 236 First Midterm (February 6, 2002) 35 points possible

You may not use calculators, cell phones, or PDAs during the exam. Partial credit is possible. Please read the entire test over before starting. Please put a box around your solutions so that the grader can find them easily.

Print your name clearly:

Please sign here:

1. (3 points) Consider the following linear programming problem:

$$\text{Maximize } 2x + 8y - z$$
$$\text{subject to}$$
$$\begin{aligned} 2x + 3y - z &= 5 \\ x - y + 8z &= 6 \\ 2y + z &= 0 \\ x, y, z &\geq 0. \end{aligned}$$

Write the problem as a linear programming problem in **standard** form.

Maximize $2x + 8y - z$

subject to

$$\begin{aligned} 2x + 3y - z &\leq 5 \\ -2x - 3y + z &\leq -5 \\ x - y + 8z &\leq 6 \\ -x + y - 8z &\leq -6 \\ 2y + z &\leq 0 \\ -2y - z &\leq 0 \end{aligned}$$

$x, y, z \geq 0$

2. (8 points) The Soft Suds Brewing and Bottling Company, because of faulty planning, was not prepared for the Operations Research Department. There was to be a big party at Stanford University[1], and Gus Guzzler, the manager, knew that Soft Suds would be called upon to supply the refreshments. However, the raw materials required had not been ordered and could not be obtained before the party. Gus took an inventory of the available supplies and found the following:

Malt	80 units
Hops	60 units
Yeast	75 units

Soft suds produces two types of pick-me-ups: light beer and dark beer, with the following specifications:

	Requirement per gallon		
	Malt	Hops	Yeast
Light beer	8	4	6
Dark beer	4	6	6

The light beer brings \$1.00/gallon profit, the dark beer \$1.50/gallon profit. Knowing the O.R. department will buy whatever is made available, formulate the linear program Gus must solve to maximize his profits, and solve it graphically. Be sure to define all your variables.

x = gallons light beer
y = gallons dark beer

units of malt used: $8x + 4y \le 80$

" hops " : $4x + 6y \le 60$

" yeast " : $6x + 6y \le 75$

maximize $x + \frac{3}{2}y$

[1]This problem is taken from Dantzig's text book. He's the inventor of the simplex method and is an O.R. professor at Stanford.

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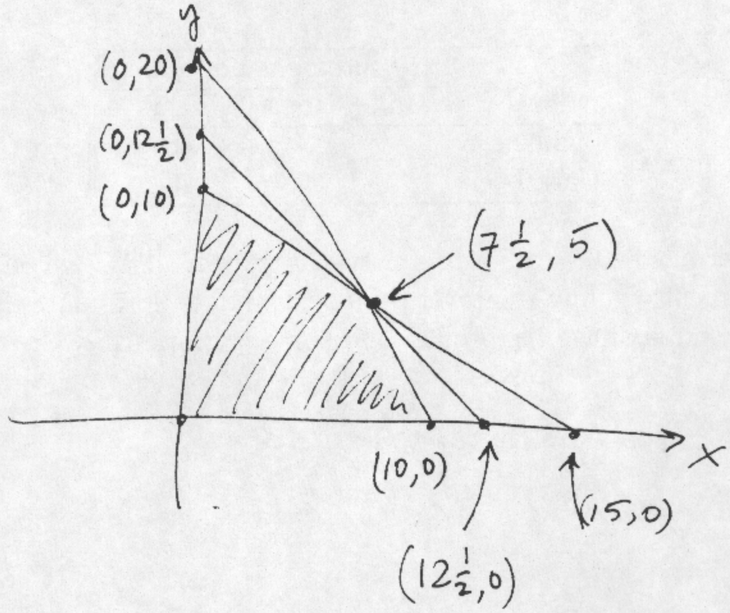
$$8x + 4y \le 80$$

$$4x + 6y \le 60$$

$$6x + 6y \le 75$$



$$4x + 6y = 60$$
$$6x + 6y = 75$$

$$2x = 15$$
$$x = 7\tfrac{1}{2}$$
$$y = 5$$

4 extreme points

$(0,0) \longrightarrow \vec{c}^{\,T} \vec{x} = 0$

$(10,0) \longrightarrow \quad = 10$

$(0,10) \longrightarrow \quad = 15$

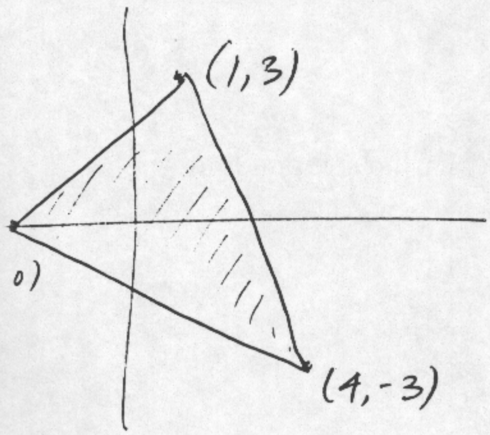
$(7\tfrac{1}{2},5) \longrightarrow \quad = 15$

so the max is achieved at all points on the edge connecting $(0,10)$ and $(7\tfrac{1}{2},5)$

3. Consider the convex region shown below.

   a. (3 points) Find a 3×2 matrix A and a vector $\vec{b}$ so that $A\vec{x} \le \vec{b}$ describes this region. **Note: Choose A and $\vec{b}$ so that all of the entries of $\vec{b}$ are either** 1 **or** -1.

   b. (3 points) Find a 4×2 matrix A and a vector $\vec{b}$ so that $A\vec{x} \le \vec{b}$ describes this region. **Note: Choose A and $\vec{b}$ so that all of the entries of $\vec{b}$ are either** 1 **or** -1.



top: $y = \frac{3-0}{1+2}(x-1)+3$

$= x+2$

right: $y = \frac{-3-3}{4-1}(x-1)+3$

$= -2x+5$

bottom: $y = \frac{-3-0}{4+2}(x-4)-3$

$= -\frac{1}{2}x - 1$

$$\begin{cases} y \le x+2 \\ y \le -2x+5 \\ y \ge -\frac{x}{2} - 1 \end{cases} \Rightarrow \begin{cases} -x+y \le 2 \\ 2x+y \le 5 \\ -\frac{x}{2} - y \le 1 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} -\frac{1}{2}x + \frac{1}{2}y \le 1 \\ \frac{2}{5}x + \frac{1}{5}y \le 1 \\ -\frac{1}{2}x - y \le 1 \end{cases}$$

so

a) $\begin{pmatrix} -1/2 & 1/2 \\ 2/5 & 1/5 \\ -1/2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \le \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

b) $\begin{pmatrix} -1/2 & 1/2 \\ 2/5 & 1/5 \\ -1/2 & -1 \\ 1/10 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \le \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

note: many different answers to part b!

4. Consider the two LP problems with two unknowns ($\vec{x} = (x_1, x_2)$):

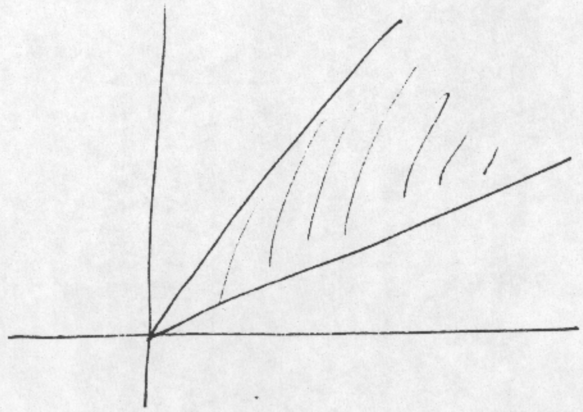
Problem 1:

$$\text{Maximize } \vec{c}^T\vec{x}$$
$$\text{subject to } A\vec{x} \leq \vec{b}$$

Problem 2:

$$\text{Minimize } \vec{c}^T\vec{x}$$
$$\text{subject to } A\vec{x} \leq \vec{b}$$

a. (3 points) Give an example of A. $\vec{b}$. and $\vec{c}$ so that Problem 1 has no solution, but Problem 2 has a solution. Explain your choice graphically.



$$\begin{cases} y \leq x \\ y \geq \frac{x}{2} \end{cases}$$

$$-x + y \leq 0$$
$$+\frac{x}{2} - y \leq 0$$

$$A = \begin{pmatrix} -1 & 1 \\ 1/2 & -1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

if $\vec{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then $\vec{c}^T\vec{x} = x + y$

will have minimum (at 0,0)
but as $x \to \infty$ or $y \to \infty$ you'll
have $\vec{c}^T\vec{x} \to \infty$ so no maximum.

b. (3 points) Give an example of A. $\vec{b}$. and $\vec{c}$ so neither Problem 1 nor Problem 2 has a solution. Explain your choice graphically.

Same A and $\vec{b}$.

take $\vec{c} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

then $\vec{c}^T \vec{x} = -x + y$

level sets are $y = x + \alpha$

and

whatever value α is, the level sets will intersect the feasible region.

So $\vec{c}^T \vec{x}$ will take all values α ($\alpha \to -\infty$ and $\alpha \to +\infty$ okay).

So no max and no min.

as $x \to \infty$ $\vec{c}^T \vec{x} \to -\infty$

as $y \to \infty$ $\vec{c}^T \vec{x} \to +\infty$

5. (6 points) Find two basic feasible solutions of

$$A\vec{x} = \vec{b}$$

where

$$A = \begin{pmatrix} 3 & 0 & 8 & 1 & 0 & -1 \\ 2 & 1 & 0 & 0 & 0 & 2 \\ 4 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\vec{b}^T = (11, 5, 4).$$

there are $\binom{6}{3} = 20$ 3x3 matrices to try and invert to find feasible solutions

four popular choices!

$$\begin{pmatrix} 0 & 8 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 11 \\ 5 \\ 4 \end{pmatrix} \to \begin{pmatrix} 0 \\ 5 \\ -4 \\ 43 \\ 0 \\ 0 \end{pmatrix}$$ not feasible!

$$\begin{pmatrix} 0 & 8 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} 11 \\ 5 \\ 4 \end{pmatrix} \to \begin{pmatrix} 0 \\ 5 \\ 1\,3/8 \\ 0 \\ 5\,3/8 \\ 0 \end{pmatrix}$$ feasible!

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 11 \\ 5 \\ 4 \end{pmatrix} \to \begin{pmatrix} 0 \\ 5 \\ 0 \\ 11 \\ 4 \\ 0 \end{pmatrix}$$ feasible!

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 11 \\ 5 \\ 4 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 13\,1/2 \\ 1\,1/2 \\ 2\,1/2 \end{pmatrix}$$ feasible!

6. (6 points) Consider the linear programming problem

$$\text{Maximize } \vec{c}^T\vec{x}$$
$$\text{subject to } A\vec{x} \leq \vec{b}.$$

Show that if there are two distinct points $\vec{x}_1$ and $\vec{x}_2$ where the objective function achieves the value 3 then there are infinitely points where the value 3 is achieved. *Note: the points being "distinct" just means $\vec{x}_1 \neq \vec{x}_2$.*

know $\vec{c}^T\vec{x}_1 = 3$
$\vec{c}^T\vec{x}_2 = 3$

let $\vec{y} = \lambda\vec{x}_1 + (1-\lambda)\vec{x}_2$

then $\vec{c}^T\vec{y} = \lambda\vec{c}^T\vec{x}_1 + (1-\lambda)\vec{c}^T\vec{x}_2$

$= \lambda \cdot 3 + (1-\lambda)3 = 3$

So any point on the line that contains $\vec{x}_1$ and $\vec{x}_2$ will have $\vec{c}^T\vec{x} = 3$.

Note: it had nothing to do with the number 3.

It's just that $\vec{c}^T\vec{x} = \alpha$ describes a hyperplane. Which contains infinitely many points and any line connecting two of them.