APM 236 Final Exam (May 1, 2002) 100 points possible

You may not use calculators, cell phones, or PDAs during the exam. Partial credit will be given for partially correct work. Please read through the entire test before starting, and take note of how many points each question is worth. Please put a box around your solutions so that the grader may find them easily.

Family name:

Given name(s):

Answer key

Please sign here:

Student ID number:

Problem 1:	/10
Problem 2:	/25
Problem 3:	/15
Problem 4:	/10
Problem 5:	/10
Problem 6:	/10
Problem 7:	/10
Problem 8:	/10
Total:	/100

1. a. (5 points) Use graphical methods to solve the problem:

Maximize $\vec{c}^T\vec{x}$ subject to

$$A\vec{x} \leq \vec{b}, \qquad \vec{x} \geq 0,$$

where

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \vec{c} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}.$$

__LP:__

maximize $2x_1 - 2x_2$

subject to $-x_1 + x_2 \leq 1$

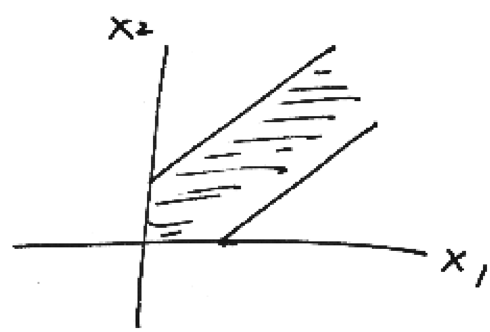
$x_1 - x_2 \leq 1$

$x_1, x_2 \geq 0$

since $-x_1 + x_2 \leq 1 \Rightarrow x_2 \leq x_1 + 1$

since $x_1 - x_2 \leq 1 \Rightarrow x_1 - 1 \leq x_2$

so feasible region is



the level sets of the objective function are

$2x_1 - 2x_2 = k \Rightarrow 2x_1 - k = 2x_2 \Rightarrow x_2 = x_1 - k/2.$

these are lines parallel to the two lines that bound the feasible region. So the maximum and minimum are achieved, both at infinitely many points. $k = 2$ at $y = x - 1$, $k = -2$ at $y = x + 1$.

$$\boxed{\text{max value} = 2 \text{ achieved at } y = x - 1,\ x \geq 0}$$

b. (5 points) State the dual problem and use graphical methods to solve it.

the dual problem is

minimize $w_1 + w_2$

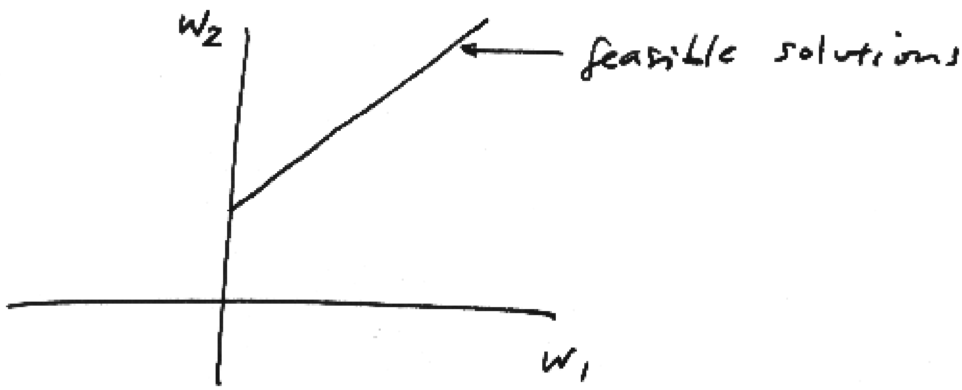
subject to

$$-w_1 + w_2 \geq 2$$
$$w_1 - w_2 \geq -2$$

$w_1, w_2 \geq 0$

The constraint set can be written as

$$\begin{array}{l} -w_1 + w_2 \geq 2 \\ -w_1 + w_2 \leq 2 \end{array} \Rightarrow -w_1 + w_2 = 2$$



$\boxed{\text{the minimum of } w_1 + w_2 \text{ occurs at } w_1 = 0 \quad w_2 = 2, \text{ min. value} = 2}$

2. (25 points) Find the optimal solution (x_1, x_2, x_3) of the following problem:

Minimize $2x_1 - x_2 + 5x_3$ subject to

$$\begin{array}{rcrcrcl} x_1 & & & + & x_3 & \geq & 2 \\ 2x_1 & + & x_2 & & & = & 2 \\ 8x_1 & + & 3x_2 & + & 2x_3 & = & 10 \end{array}$$

where $x_1 \geq 0$, $x_2 \leq 0$, and x_3 is unrestricted.

This is a 25 point problem. If you're having problems or are running out of time, for 15 points you may solve the problem with the constraints $x_1, x_2, x_3 \geq 0$ instead.

I want to use the simplex method. This requires that the variables be ≥ 0. So since $x_2 \leq 0$, I'll replace x_2 by $-x_2$. Also, I'll replace x_3 by $x_3 - x_4$. I have to be careful to translate the optimal solution I find back to the original x_1 (≥ 0), x_2 (≤ 0), and x_3 (unrestricted)!

Minimize $2x_1 + x_2 + 5x_3 - 5x_4$
subject to

$$\begin{array}{rl} x_1 \qquad + x_3 - x_4 & \geq 2 \\ 2x_1 - x_2 \qquad\qquad & = 2 \\ 8x_1 - 3x_2 + 2x_3 - 2x_4 & = 10 \end{array}$$

$x_1, x_2, x_3, x_4 \geq 0$

Introduce slack variable x_5 into constraint 1.
Introduce <u>three</u> artificial variables: y_1, y_2, y_3

Minimize $2x_1 + x_2 + 5x_3 - 5x_4$

subject to

$$\begin{array}{llllllll} x_1 & & +x_3 & -x_4 & -x_5 & +y_1 & & = 2 \\ 2x_1 & -x_2 & & & & & +y_2 & = 2 \\ 8x_1 & -3x_2 & +2x_3 & -2x_4 & & & +y_3 & = 10 \end{array}$$

To begin phase 1, I want to maximize a different objective function: $-y_1 - y_2 - y_3$. I need to write it in terms of the nonbasic variables x_1, x_2, x_3, x_4, x_5. I use the equations to do this.

e1 $\Rightarrow$ $-y_1 = x_1 + x_3 - x_4 - x_5 - 2$

e2 $\Rightarrow$ $-y_2 = 2x_1 - x_2 - 2$

e3 $\Rightarrow$ $-y_3 = 8x_1 - 3x_2 + 2x_3 - 2x_4 - 10$

So the objective function I'm maximizing is

$$11x_1 - 4x_2 + 3x_3 - 3x_4 - x_5 - 14$$

initial tableau:

	x_1 ↓	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
y_1	1	0	1	-1	-1	1	0	0	2
← y_2	2	-1	0	0	0	0	1	0	2
y_3	8	-3	2	-2	0	0	0	1	10
	-11	4	-3	3	1	0	0	0	-14

x_1 incoming, y_2 departing.

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
← y_1	0	$1/2$	1	-1	-1	1	$-1/2$	0	1
x_1	1	$-1/2$	0	0	0	0	$1/2$	0	1
y_3	0	1	2	-2	0	0	-4	1	2
	0	$-3/2$	-3	3	1	0	$1/2$	0	-3

$4 - \frac{11}{2} = \frac{8}{2} - \frac{11}{2} = -3/2$

x_3 entering, y_1 departing

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
x_3	0	$1/2$	1	-1	-1	1	$-1/2$	0	1
x_1	1	$-1/2$	0	0	0	0	$1/2$	0	1
← y_3	0	0	0	0	2	-2	-3	1	0
	0	0	0	0	-2	3	4	0	0

x_5 entering, y_3 departing

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
x_3	0	$1/2$	1	-1	0	0	-2	$1/2$	1
x_1	1	$-1/2$	0	0	0	0	$1/2$	0	1
x_5	0	0	0	0	1	-1	$-3/2$	$1/2$	0
	0	0	0	0	0	1	1	1	0

phase 1 terminates!

$x_1 = 1$, $x_3 = 1$, $x_5 = 0$ are the basic variables.

check:
- constr 1: $x_1 + x_3 - x_4 - x_5 = 1 + 1 = 2$ ✓
- constr 2: $2x_1 - x_2 = 2 \cdot 1 = 2$ ✓
- constr 3: $8x_1 - 3x_2 + 2x_3 - 2x_4 = 8 + 2 = 10$ ✓

obj function: $2x_1 + x_2 + 5x_3 - 5x_4 = 7$

Now we begin phase 2. we are <u>maximizing</u>

$$-2x_1 - x_2 - 5x_3 + 5x_4$$

the basic variables are $x_1, x_3,$ and x_5. So we need to write the obj. function in terms of the nonbasic variables x_2 & x_4. We use the 3 eqns in the last tableau:

e1: $\frac{1}{2}x_2 + x_3 - x_4 = 1 \rightarrow x_3 = -\frac{1}{2}x_2 + x_4 + 1$

e2: $x_1 - \frac{1}{2}x_2 = 1 \rightarrow x_1 = \frac{1}{2}x_2 + 1$

e3: $x_5 = 0$

obj. function:

$$= -2\left(\frac{1}{2}x_2 + 1\right) - x_2 - 5\left(-\frac{1}{2}x_2 + x_4 + 1\right) + 5x_4$$

$$= -x_2 - 2 - x_2 + 5x_2 - 5x_4 - 5 + 5x_4$$

$$= 3x_2 - 7$$

	x_1	x_2	x_3	x_4	x_5	
x_3	0	$\frac{1}{2}$	1	-1	0	1
x_1	1	$-\frac{1}{2}$	0	0	0	1
x_5	0	0	0	0	1	0
	0	-3	0	0	0	7

x_2 entering, x_3 departing

	x_1	x_2	x_3	x_4	x_5	
x_2	0	1	2	-2	0	2
x_1	1	0	1	-1	0	2
x_5	0	0	0	0	1	0
	0	0	6	-6	0	13

x_4 entering, no departing variable.

the problem is unbounded $\Rightarrow$ no finite solution!

<u>Note</u>: had the constraints been $x_1, x_2, x_3 \geq 0$ then the optimal solution would have been $x_1 = 1$, $x_2 = 0$, $x_3 = 1$ with max. value = 7.

3. Consider the primal problem:

Minimize $4x_1 + 2x_2 + x_3$ subject to

$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & + & x_3 & \geq & 1 \\ & & x_2 & - & x_3 & = & 5 \\ x_1 & + & x_2 & - & 2x_3 & \leq & 8 \end{array}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and x_3 is unrestricted.

a. (3 points) State the dual problem

I know how to find the dual of
"maximize $\vec{c}^T\vec{x}$
subject to $A\vec{x} \leq \vec{b}$
$\vec{x} \geq 0$ "

So I write the problem in that form.
I replace x_3 with $x_3 - x_4$ to do this.

<u>primal:</u> maximize $-4x_1 - 2x_2 - x_3 + x_4$
subject to

$$\begin{aligned} -2x_1 - x_2 - x_3 + x_4 &\leq -1 \\ x_2 - x_3 + x_4 &= 5 \\ x_1 + x_2 - 2x_3 + 2x_4 &\leq 8 \end{aligned}$$

$x_1, x_2, x_3, x_4 \geq 0$

I break the 2nd constraint into 2 inequalities:

<u>primal:</u> maximize $-4x_1 - 2x_2 - x_3 + x_4$
subject to:

$$\begin{aligned} -2x_1 - x_2 - x_3 + x_4 &\leq -1 \\ x_2 - x_3 + x_4 &\leq 5 \\ -x_2 + x_3 - x_4 &\leq -5 \\ x_1 + x_2 - 2x_3 + 2x_4 &\leq 8 \end{aligned}$$

$x_1, x_2, x_3, x_4 \geq 0$

dual: minimize $-w_1+5w_2-5w_3+8w_4$

subject to

$$
\begin{aligned}
-2w_1 \qquad\qquad\qquad + w_4 &\geq -4 \\
-w_1 + w_2 - w_3 + w_4 &\geq -2 \\
-w_1 - w_2 + w_3 - 2w_4 &\geq -1 \\
w_1 + w_2 - w_3 + 2w_4 &\geq 1
\end{aligned}
$$

$w_1, w_2, w_3, w_4 \geq 0$

the 3rd & fourth constraints become an equality. and the variables w_2, w_3 always appear as opposite pairs, so I introduce $u = -w_2 + w_3$

dual: minimize $-w_1 - 5u + 8w_4$

subject to

$$
\begin{aligned}
-2w_1 \qquad + w_4 &\geq -4 \\
-w_1 - u + w_4 &\geq -2 \\
-w_1 + u - 2w_4 &= -1
\end{aligned}
$$

$w_1, w_4 \geq 0$, u unrestricted.

writing it all in terms of w_1, w_2, w_3:

dual: maximize $w_1 + 5w_2 - 8w_3$

subject to

$$
\begin{aligned}
-2w_1 \qquad + w_3 &\geq -4 \\
-w_1 - w_2 + w_3 &\geq -2 \\
-w_1 + w_2 - 2w_3 &= -1
\end{aligned}
$$

$w_1, w_3 \geq 0$ w_2 unrestricted

b. (12 points) An optimal solution of the primal problem is $x_1 = 0$, $x_2 = 3$, $x_3 = -2$. Use this and complementary slackness to solve the dual problem.

primal: obj. function value $= 4 \cdot 0 + 2 \cdot 3 + (-2) = 4$

test for slackness:

constr 1: $2 \cdot 0 + 3 + (-2) = 1 \geq 1$ no slack

constr 2: $3 - (-2) = 5 = 5$ no slack (obviously)

constr 3: $0 + 3 - 2(-2) = 7 < 8$ slack.

this tells us that the slack variables of the primal problem are

$$x_1' = 0 \quad x_2' = 0 \quad x_3' > 0$$

since

$$x_1' w_1 = 0$$
$$x_2' w_2 = 0$$
$$x_3' w_3 = 0$$

by compl. slackness, this gives us $\boxed{w_3 = 0}$

on the other hand, complementary slackness also tells us

$$x_1 \, w_1' = 0$$
$$x_2 \, w_2' = 0 \quad \Rightarrow \quad w_2' = 0$$
$$x_3 \, w_3' = 0 \qquad\quad w_3' = 0$$

$\Rightarrow$ no slack in 2nd & 3rd constraints of dual problem.

So at the optimal solution of dual problem, we know

$$w_1 + 5w_2 - 8w_3 = 4 \leftarrow \text{obj. function}$$

$$w_3 = 0 \leftarrow \text{since slack in 3rd constr. of primal}$$

$$-w_1 - w_2 + w_3 = -2 \leftarrow \text{since } x_2 \neq 0$$

$$-w_1 + w_2 - 2w_3 = -1 \leftarrow \text{since } x_3 \neq 0 \text{ and since it was an equality all along.}$$

$$\begin{aligned} -w_1 - w_2 &= -2 \\ -w_1 + w_2 &= -1 \end{aligned} \Rightarrow -2w_1 = -3$$

$$\Rightarrow \boxed{w_1 = 3/2} \qquad \boxed{w_3 = 0}$$

$$-w_1 + w_2 = -1 \Rightarrow -\frac{3}{2} + w_2 = -1$$

$$\Rightarrow \boxed{w_2 = 1/2}$$

check this is consistent with the first equation

$$\frac{3}{2} + 5\left(\frac{1}{2}\right) - 8(0) \stackrel{?}{=} 4 \quad \checkmark$$

solution of dual problem:

$$w_1 = 3/2$$
$$w_2 = 1/2$$
$$w_3 = 0$$

4. (10 points)

Lyosha, Yael, Yi Li, and Vinh have been hired by a restaurant. The manager needs to assign four jobs: dishwasher, cook, busboy, and waiter. Yi Li insists on \$7/hour to be a busboy or a dishwasher, \$10/hour to be a waiter, and \$11/hour to be a cook. Lyosha has more experience than Yi Li, so he is demanding \$2 more per hour than she does, with one exception. He loves to cook, so he's willing to do that for \$10/hour. Yael doesn't care what job she's given; she just wants \$9/hour. Vinh has the least experience and so would accept \$1 less per hour than Yi Li, with one exception: He has delicate hands and would wash dishes only for \$12/hour.

The manager wants to assign the jobs in a way that would spend the least possible amount on wages. Who should get what job? How much per hour will the restaurant be paying for these four workers?

In case you've never worked in a restaurant: busboys and waiters may be male or female.

$$C = \begin{array}{c} \begin{array}{cccc} \text{dish.} & \text{cook.} & \text{busb.} & \text{waiter} \end{array} \\ \begin{pmatrix} 9 & 10 & 9 & 12 \\ 9 & 9 & 9 & 9 \\ 7 & 11 & 7 & 10 \\ 12 & 10 & 6 & 9 \end{pmatrix} \end{array} \begin{array}{l} \text{Lyosha} \\ \text{Yael} \\ \text{Yi Li} \\ \text{Vinh} \end{array}$$

step 1: subtract minimum from each row

$$C' = \begin{pmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 3 \\ 6 & 4 & 0 & 3 \end{pmatrix}$$

stop since there's a 0 in each row & column.

$$C' = \begin{pmatrix} 0^* & 1 & 0 & 3 \\ 0 & 0^* & 0 & 0 \\ 0 & 4 & 0^* & 3 \\ 6 & 4 & 0 & 3 \end{pmatrix}$$

4th row has no assigned 0.

(4,3) 0
(3,3) 0^*
(3,1) 0
(1,1) 0^*

flipping won't help.
cols 1 & 3 are necessary.
row 2 is necessary.

$$\begin{matrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 3 \\ 6 & 4 & 0 & 3 \end{matrix} \rightarrow C' = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 2 \\ 6 & 3 & 0 & 2 \end{pmatrix}$$

assign 0's: $\begin{pmatrix} 0^* & 0 & 0 & 2 \\ 1 & 0^* & 1 & 0 \\ 0 & 3 & 0^* & 2 \\ 6 & 3 & 0 & 2 \end{pmatrix}$

(4,3) 0
(3,3) 0^*
(3,1) 0
(1,1) 0^*
(1,2) 0
(2,2) 0^*
(2,4) 0

flip!
and done

$$\begin{pmatrix} 0 & 0^* & 0 & 2 \\ 1 & 0 & 1 & 0^* \\ 0^* & 3 & 0 & 2 \\ 6 & 3 & 0^* & 2 \end{pmatrix}$$

Lyosha = cook
Yael = waiter
Yi Li = dishwasher
Vinh = busboy

cost = 10 + 9 + 7 + 6 = 32 \$/hour

5. (10 points) Solve the assignment problem with the given cost matrix. Show your work. Also, give the cost of the optimal solution.

Do not solve the problem by inspection! At each step, say what you're doing and make it clear to the grader that you're using the Hungarian algorithm of §5.2.

$$C = \begin{pmatrix} 3 & 2 & 7 & 4 & 8 \\ 5 & 4 & 3 & 8 & 5 \\ 3 & 7 & 9 & 1 & 2 \\ 4 & 2 & 6 & 5 & 7 \\ 2 & 8 & 4 & 6 & 6 \end{pmatrix}$$

subtract min from all rows.

$$C' = \begin{matrix} 1 & 0 & 5 & 2 & 6 \\ 2 & 1 & 0 & 5 & 2 \\ 2 & 6 & 8 & 0 & 1 \\ 2 & 0 & 4 & 3 & 5 \\ 0 & 6 & 2 & 4 & 4 \end{matrix}$$

subtract min from all columns

$$C' = \begin{pmatrix} 1 & 0 & 5 & 2 & 5 \\ 2 & 1 & 0 & 5 & 1 \\ 2 & 6 & 8 & 0 & 0 \\ 2 & 0 & 4 & 3 & 4 \\ 0 & 6 & 2 & 4 & 3 \end{pmatrix}$$

try assigning by the rule

$$\begin{matrix} 1 & 0^* & 5 & 2 & 5 \\ 2 & 1 & 0^* & 5 & 1 \\ 2 & 6 & 8 & 0^* & 0 \\ 2 & 0 & 4 & 3 & 4 \\ 0^* & 6 & 2 & 4 & 3 \end{matrix}$$

4th row has no assigned 0. Look for a change of assignments.

$(4,2)$ 0

$(1,2)$ 0^*

none poss. col 2 is necc.

→ rows 2, 3, 5 are necc.

$$\begin{matrix} 1 & 0 & 5 & 2 & 5 \\ 2 & 1 & 0 & 5 & 1 \\ 2 & 6 & 8 & 0 & 0 \\ 2 & 0 & 4 & 3 & 4 \\ 0 & 9 & 2 & 4 & 3 \end{matrix}$$

$$C' = \begin{matrix} 0^* & 0 & 4 & 1 & 4 \\ 2 & 2 & 0^* & 5 & 1 \\ 2 & 7 & 8 & 0^* & 0 \\ 1 & 0^* & 3 & 2 & 3 \\ 0 & 7 & 2 & 4 & 3 \end{matrix}$$

try assigning 0's..

try permuting them...

$(5,1)$ 0
$(1,1)$ 0^*
$(1,2)$ 0
$(4,2)$ 0^*

$\Rightarrow$ cols 1 & 2 are nec.
$\Rightarrow$ rows 2 & 3 are nec.

$$\begin{matrix} 0 & 0 & 4 & 1 & 4 \\ 2 & 2 & 0 & 5 & 1 \\ 2 & 7 & 8 & 0 & 0 \\ 1 & 0 & 3 & 2 & 3 \\ 0 & 7 & 2 & 4 & 3 \end{matrix} \rightarrow C' = \begin{matrix} 0^* & 0 & 3 & 0 & 3 \\ 3 & 3 & 0^* & 5 & 1 \\ 3 & 8 & 8 & 0^* & 0 \\ 1 & 0^* & 2 & 1 & 2 \\ 0 & 7 & 1 & 3 & 2 \end{matrix}$$

try permuting..

$(5,1)$ 0
$(1,1)$ 0^*
$(1,4)$ 0
$(3,4)$ 0^*
$(3,5)$ 0

flip!

$$C' = \begin{pmatrix} 0 & 0 & 3 & 0^* & 3 \\ 3 & 3 & 0^* & 5 & 1 \\ 3 & 8 & 8 & 0 & 0^* \\ 1 & 0^* & 2 & 1 & 2 \\ 0^* & 7 & 1 & 3 & 2 \end{pmatrix}$$

$$\boxed{x_{14}=1,\ x_{23}=1,\ x_{35}=1,\ x_{42}=1,\ x_{51}=1}$$

cost $= 4+3+2+2+2 = \boxed{13}$

6. (10 points) Consider the following transportation problem:

Minimize $2x_{11} + 2x_{12} + x_{13} + x_{21} + x_{22} + 3x_{23} + 2x_{31} + 2x_{32} + x_{33}$
Subject to

$$
\begin{aligned}
x_{11} + x_{12} + x_{13} &= 25 \\
x_{21} + x_{22} + x_{23} &= 25 \\
x_{31} + x_{32} + x_{33} &= 10 \\
x_{11} + x_{21} + x_{31} &= 10 \\
x_{12} + x_{22} + x_{32} &= 20 \\
x_{13} + x_{23} + x_{33} &= 30
\end{aligned}
$$

where $x_{ij} \geq 0$ for $1 \leq i \leq 3$, $1 \leq j \leq 3$.

Find an optimal solution. What is its cost?

$\boxed{2}$ 0	$\boxed{2}$ 0	$\boxed{1}$ 25	25
$\boxed{1}$ 5	$\boxed{1}$ 20	$\boxed{3}$ 0	25
$\boxed{2}$ 5	$\boxed{2}$ 0	$\boxed{1}$ 5	10
10	20	30	

$x_{13} = 25 \quad x_{21} = 5 \quad x_{22} = 20 \quad x_{31} = 5 \quad x_{33} = 5$

cost $= 25 + 5 + 20 + 10 + 5 = 65$

is it optimal?

$$
\begin{aligned}
x_{13} &: V_1 + W_3 = 1 \\
x_{21} &: V_2 + W_1 = 1 \\
x_{22} &: V_2 + W_2 = 1 \\
x_{31} &: V_3 + W_1 = 2 \\
x_{33} &: V_3 + W_3 = 1
\end{aligned}
\rightarrow
\begin{aligned}
V_1 &= 0 \\
V_2 &= -1 \\
V_3 &= 0 \\
W_1 &= 2 \\
W_2 &= 2 \\
W_3 &= 1
\end{aligned}
$$

obj. row

$$
\left.
\begin{aligned}
x_{11} &: V_1 + W_1 - C_{11} = 0 + 2 - 2 = 0 \\
x_{12} &: V_1 + W_2 - C_{12} = 0 + 2 - 2 = 0 \\
x_{23} &: V_2 + W_3 - C_{23} = -1 + 1 - 3 = -3 \\
x_{32} &: V_3 + W_2 - C_{32} = 0 + 2 - 2 = 0
\end{aligned}
\right\}
$$

none are > 0 so optimal.

7. Consider the following transportation problem:

Minimize $2x_{11} + 1x_{12} + 3x_{13} + 2x_{21} + 3x_{22} + 2x_{23} + 4x_{31} + x_{32} + 3x_{33}$
Subject to

$$\begin{aligned}
x_{11} + x_{12} + x_{13} &= 10 \\
x_{21} + x_{22} + x_{23} &= 20 \\
x_{31} + x_{32} + x_{33} &= 25 \\
x_{11} + x_{21} + x_{31} &= 15 \\
x_{12} + x_{22} + x_{32} &= 20 \\
x_{13} + x_{23} + x_{33} &= 20
\end{aligned}$$

where $x_{ij} \geq 0$ for $1 \leq i \leq 3$, $1 \leq j \leq 3$.

At one point in the minimization process, the basic variables are x_{11}, x_{21}, x_{22}, x_{31}, and x_{33}.

a. (2 points) Fill in the transportation tableau below and compute the current value of the objective function.

$\boxed{2}$ 10	$\boxed{1}$ 0	$\boxed{3}$ 0	10
$\boxed{2}$ 0	$\boxed{3}$ 20	$\boxed{2}$ 0	20
$\boxed{4}$ 5	$\boxed{1}$ 0	$\boxed{3}$ 20	25
15	20	20	

Cost $= 2 \cdot 10 + 3 \cdot 20 + 4 \cdot 5 + 3 \cdot 20$

$= 20 + 60 + 20 + 60$

$= 160$

b. (5 points) Complete the following simplex tableau

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}	
x_{11}	1	1	1	0	0	0	0	0	0	10
x_{21}	0	-1	0	1	0	1	0	-1	0	0
x_{22}	0	1	0	0	1	0	0	1	0	20
x_{31}	0	0	-1	0	0	-1	1	1	0	5
x_{33}	0	0	1	0	0	1	0	0	1	20
	0	2	-2	0	0	-1	0	4	0	160

X_{12} entering:

-1	+1	
+1	-1	

cost $= 1 - 2 + 2 - 3 = -2$

X_{13} entering:

-1		+1
+1		-1

cost $= 3 - 2 + 4 - 3 = 2$

X_{23} entering:

-1		+1
+1		-1

cost $= 2 - 2 + 4 - 3 = 1$

X_{32} entering:

+1	-1	
-1	+1	

cost $= 1 - 3 + 2 - 4 = -4$

c. (3 points) Choose an entering variable. What is the departing variable? Fill in the resulting transportation tableau below and find the new value of the objective function.

if x_{12} enters then x_{11} departs

$\boxed{2}$	$\boxed{1}$ 10	$\boxed{3}$	10
$\boxed{2}$ 10	$\boxed{3}$ 10	$\boxed{2}$	20
$\boxed{4}$ 5	$\boxed{1}$	$\boxed{3}$ 20	25
15	20	20	

cost $= 160 - 10 \cdot 2 = 140$

if x_{32} enters then x_{31} departs

$\boxed{2}$ 10	$\boxed{1}$	$\boxed{3}$	10
$\boxed{2}$ 5	$\boxed{3}$ 15	$\boxed{2}$	20
$\boxed{4}$	$\boxed{1}$ 5	$\boxed{3}$ 20	25
15	20	20	

cost $= 160 - 5 \cdot 4$

$= 140$

8. Consider the following simplex tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
x_3	1	0	1	−2	0	6	0	0	2
x_2	−2	1	0	2	0	−3	−2	0	0
x_5	2	0	0	0	1	6	−3	0	2
x_8	0	0	0	4	0	−1	9	1	0
	−2	0	0	1	0	3	−1	0	9

a. (1 point) What are the basic variables?

$x_3 \quad x_2 \quad x_5 \quad x_8$

b. (1 point) What are the nonbasic variables?

$x_1 \quad x_4 \quad x_6 \quad x_7$

c. (6 points) For each nonbasic variable, fill in the blanks in the following: "If ___ were to be an entering variable then the departing variable(s) would be ___ and the new value of the objective function would be ___" *If you find this question confusing, look at parts d and e of this problem and see if that helps.*

if x_1 enters then x_5 departs and
the new value of obj. function would be 11

if x_4 enters then either x_2 or x_8 could depart.
the new value of the obj. function would be 9

if x_6 enters then either x_3 or x_5 could
depart. The new value of the obj. function
would be 8

if x_7 enters then x_8 departs. The new
value of the objective function would be 9

d. (1 point) If the simplex tableau arose as part of a minimization problem, what would you choose as the entering variable?

take x_6 entering.

e. (1 point) If the simplex tableau arose as part of a maximization problem, what would you choose as the entering variable?

take x_1 entering.