THE FACULTY OF ARTS AND SCIENCE
University of Toronto

FINAL EXAMINATIONS, APRIL/MAY 2002

APM236H1S
Applications of Linear Programming

Examiner: Professor M. Pugh
Duration: 2 hours

NO AIDS ALLOWED. **Total: 100 marks**

Family Name: ______________________________
(Please Print)

Given Name(s): ______________________________
(Please Print)

Please sign here: ______________________________

Student ID Number: ______________________________

You may not use calculators, cell phones, or PDAs during the exam. Partial credit will be given for partially correct work. Please read through the entire test before starting, and take note of how many points each question is worth. Please put a box around your solutions so that the grader may find them easily.

FOR MARKER'S USE ONLY	
Problem 1:	/10
Problem 2:	/25
Problem 3:	/15
Problem 4:	/10
Problem 5:	/10
Problem 6:	/10
Problem 7:	/10
Problem 8:	/10
TOTAL:	/100

1. a. (5 points) Use graphical methods to solve the problem:

Maximize $\vec{c}^T\vec{x}$ subject to

$$A\vec{x} \leq \vec{b}, \qquad \vec{x} \geq 0,$$

where

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \text{and} \quad \vec{c} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}.$$

b. (5 points) State the dual problem and use graphical methods to solve it.

2. (25 points) Find the optimal solution (x_1, x_2, x_3) of the following problem:

Minimize $2x_1 - x_2 + 5x_3$ subject to

$$\begin{array}{ccccccc} x_1 & & & + & x_3 & \geq & 2 \\ 2x_1 & + & x_2 & & & = & 2 \\ 8x_1 & + & 3x_2 & + & 2x_3 & = & 10 \end{array}$$

where $x_1 \geq 0$, $x_2 \leq 0$, and x_3 is unrestricted.

This is a 25 point problem. If you're having problems or are running out of time, for 15 points you may solve the problem with the constraints $x_1, x_2, x_3 \geq 0$ instead.

Extra page if needed.

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3. Consider the primal problem:

Minimize $4x_1 + 2x_2 + x_3$ subject to

$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & + & x_3 & \geq & 1 \\ & & x_2 & - & x_3 & = & 5 \\ x_1 & + & x_2 & - & 2x_3 & \leq & 8 \end{array}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and x_3 is unrestricted.

a. (3 points) State the dual problem

Extra page if needed.

b. (12 points) An optimal solution of the primal problem is $x_1 = 0$, $x_2 = 3$, $x_3 = -2$. Use this and complementary slackness to solve the dual problem.

Extra page if needed.

4. (10 points)

A restaurant has hired four employees:

1) Lyosha, 2) Yael, 3) Yi Li, and 4) Vinh.

The manager needs to assign four jobs:

1) dishwasher, 2) cook, 3) busboy, and 4) waiter.

Yi Li insists on \$7/hour to be a busboy or a dishwasher, \$10/hour to be a waiter, and \$11/hour to be a cook. Lyosha has more experience than Yi Li, so he is demanding \$2 more per hour than she does, with one exception. He loves to cook, so he's willing to do that for \$10/hour. Yael doesn't care what job she's given; she just wants \$9/hour. Vinh has the least experience and so would accept \$1 less per hour than Yi Li, with one exception: He has delicate hands and would wash dishes only for \$12/hour.

The manager wants to assign the jobs in a way that would spend the least possible amount on wages. Who should get what job? How much per hour will the restaurant be paying for these four workers? (Please index the workers and jobs as above. For example, C_{23} is how much Yael would be paid to be a busboy.)

In case you've never worked in a restaurant: busboys and waiters may be male or female.

Extra page if needed.

5. (10 points) Solve the assignment problem with the given cost matrix. Show your work. Also, give the cost of the optimal solution.

Do not solve the problem by inspection! At each step, say what you're doing and make it clear to the grader that you're using the Hungarian algorithm of §5.2.

$$C = \begin{pmatrix} 3 & 2 & 7 & 4 & 8 \\ 5 & 4 & 3 & 8 & 5 \\ 3 & 7 & 9 & 1 & 2 \\ 4 & 2 & 6 & 5 & 7 \\ 2 & 8 & 4 & 6 & 6 \end{pmatrix}$$

Extra page if needed.

6. (10 points) Consider the following transportation problem:

Minimize $2x_{11} + 2x_{12} + x_{13} + x_{21} + x_{22} + 3x_{23} + 2x_{31} + 2x_{32} + x_{33}$
Subject to

$$\begin{aligned}
x_{11} + x_{12} + x_{13} &= 25 \\
x_{21} + x_{22} + x_{23} &= 25 \\
x_{31} + x_{32} + x_{33} &= 10 \\
x_{11} + x_{21} + x_{31} &= 10 \\
x_{12} + x_{22} + x_{32} &= 20 \\
x_{13} + x_{23} + x_{33} &= 30
\end{aligned}$$

where $x_{ij} \geq 0$ for $1 \leq i \leq 3$, $1 \leq j \leq 3$.

Find an optimal solution. What is its cost?

7. Consider the following transportation problem:

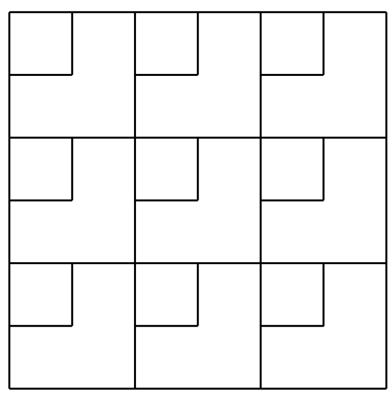
Minimize $2x_{11} + 1x_{12} + 3x_{13} + 2x_{21} + 3x_{22} + 2x_{23} + 4x_{31} + x_{32} + 3x_{33}$
Subject to

$$\begin{aligned} x_{11} + x_{12} + x_{13} &= 10 \\ x_{21} + x_{22} + x_{23} &= 20 \\ x_{31} + x_{32} + x_{33} &= 25 \\ x_{11} + x_{21} + x_{31} &= 15 \\ x_{12} + x_{22} + x_{32} &= 20 \\ x_{13} + x_{23} + x_{33} &= 20 \end{aligned}$$

where $x_{ij} \geq 0$ for $1 \leq i \leq 3$, $1 \leq j \leq 3$.

At one point in the minimization process, the basic variables are x_{11}, x_{21}, x_{22}, x_{31}, and x_{33}.

a. (2 points) Fill in the transportation tableau below and compute the current value of the objective function.



b. (5 points) Complete the following simplex tableau

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}	
x_{11}	1			0	0		0		0	10
x_{21}	0			1	0		0		0	
x_{22}	0			0	1		0		0	
x_{31}	0			0	0		1		0	
x_{33}	0			0	0		0		1	
	0			0	0		0		0	

c. (3 points) Choose an entering variable. What is the departing variable? Fill in the resulting transportation tableau below and find the new value of the objective function.

8. Consider the following simplex tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
x_3	1	0	1	-2	0	6	0	0	2
x_2	-2	1	0	2	0	-3	-2	0	0
x_5	2	0	0	0	1	6	-3	0	2
x_8	0	0	0	4	0	-1	9	1	0
	-2	0	0	1	0	3	-1	0	9

a. (1 point) What are the basic variables?

b. (1 point) What are the nonbasic variables?

c. (6 points) For each nonbasic variable, fill in the blanks in the following: "If ___ were to be an entering variable then the departing variable(s) would be ___ and the new value of the objective function would be ___" *If you find this question confusing, look at parts d and e of this problem and see if that helps.*

d. (1 point) If the simplex tableau arose as part of a minimization problem, what would you choose as the entering variable?

e. (1 point) If the simplex tableau arose as part of a maximization problem, what would you choose as the entering variable?