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NONNEGATIVE SOLUTIONS FOR A LONG-WAVE UNSTABLE THIN FILM EQUATION WITH CONVECTION*

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Abstract. We consider a nonlinear fourth-order degenerate parabolic partial differential equation that arises in modeling the dynamics of an incompressible thin liquid film on the outer surface of a rotating horizontal cylinder in the presence of gravity. The parameters involved determine a rich variety of qualitatively different flows. Depending on the initial data and the parameter values, we prove the existence of nonnegative periodic weak solutions. In addition, we prove that these solutions and their gradients cannot grow any faster than linearly in time; there cannot be a finite-time blowup. Finally, we present numerical simulations of solutions.

Key words. fourth-order degenerate parabolic equations, thin liquid films, convection, rimming flows, coating flows

AMS subject classifications. 35K65, 35K35, 35Q35, 35G25, 35B40, 35B99, 35D05, 76A20

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1. Introduction.

We consider the dynamics of a viscous incompressible fluid on the outer surface of a horizontal circular cylinder that is rotating around its axis in the presence of gravity; see Figure 1.

If the cylinder is fully coated there is only one free boundary: where the liquid meets the surrounding air. Otherwise, there is also a free boundary (or contact line) where the air and liquid meet the cylinder's surface.

The motion of the liquid film is governed by four physical effects: viscosity, gravity, surface tension, and centrifugal forces. These are reflected in the following parameters: R, the radius of the cylinder; ω, its rate of rotation (assumed constant); g, the acceleration due to gravity; ν, the kinematic viscosity; ρ, the fluid's density; and σ, the surface tension.

These parameters yield three independent dimensionless numbers: the Reynolds number $\mathrm{Re} = (R^2\omega)/\nu$, $\gamma = g/(R\omega^2)$, and the Weber number $\mathrm{We} = (\rho R^3\omega^2)/\sigma$.

We introduce the parameter $\epsilon = \bar{h}/R$, where $\bar{h}$ is the average thickness of the liquid. The following limiting regime is considered as $\epsilon \to 0$ [32, 33, 3, 29]:

$$\kappa = \mathrm{Re}\,\epsilon^2 \to 0, \quad \chi = \frac{\mathrm{Re}}{\mathrm{We}}\,\epsilon^3 \to c_1, \quad \text{and} \quad \mu = \gamma\,\mathrm{Re}\,\epsilon^2 \to c_2, \tag{1.1}$$

where c_1 and c_2 are finite and nonzero.

One can model the flow using the full three-dimensional Navier–Stokes equations with free boundaries: for $\vec{u}(x, y, z, t)$ in the region $x \in [-\pi, \pi)$, $y \in \mathbb{R}^1$, and $z \in (0, h(x, y, t))$, where x is the angular variable, y is the axial variable, and $h(x, y, t)$ is the thickness of the fluid above the point (x, y) on the surface of the cylinder at time t. This has been done by Pukhnachov [32, Theorem 1], who proved the existence and

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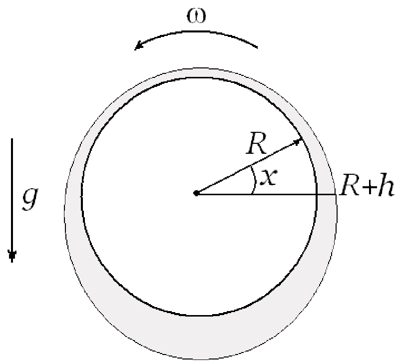

FIG. 1. *Liquid film on the outer surface of a rotating horizontal cylinder in the presence of gravity.*

uniqueness of fully coating steady states (no contact line is present) if γ is not too large. We know of no results for the affiliated initial value problem.

In this physical regime, if one also makes a long-wave approximation (the thickness of the coating fluid is smaller than the radius of the cylinder), and if one further assumes that the rotation rate is low (or the viscosity is large), then the three-dimensional Navier–Stokes equations with free boundary can be approximated by a fourth-order degenerate partial differential equation (PDE) for the film thickness $h(x, y, t)$. This is done by averaging the fluid flow in the direction normal to the cylinder [32, 33]. If one further assumes that the flow is independent of the axial variable, y, then this results in a PDE in one dimension for $h(x, t)$.

In his pioneering 1977 article about syrup rings on a rotating roller, Moffatt neglected the effect of surface tension (i.e., $\mathrm{We}^{-1} = 0 = \chi$), assumed the flow was uniform in the axial variable, and derived [29] the following model for the thin film thickness:

$$h_t + \left(h - \tfrac{\mu}{3} h^3 \cos(x)\right)_x = 0, \tag{1.2}$$

where μ is given in (1.1) and

$$x \in [-\pi, \pi], \quad t > 0, \quad h \text{ is } 2\pi\text{-periodic in } x.$$

Pukhnachov's 1977 article [32] gives the first model that takes into account surface tension:

$$h_t + (h - \tfrac{\mu}{3} h^3 \cos(x))_x + \tfrac{\chi}{3} \left(h^3 (h_x + h_{xxx})\right)_x = 0, \tag{1.3}$$

where μ and χ are given in (1.1) and

$$x \in [-\pi, \pi], \quad t > 0, \quad h \text{ is } 2\pi\text{-periodic in } x.$$

This model assumes a no-slip boundary condition at the liquid/solid interface. For a solution to (1.2) or (1.3) to be physically relevant, either h is strictly positive (the cylinder is fully coated) or h is nonnegative (the cylinder is wet in some region and dry in others).

Weidner, Schwartz, and Eres [44] present modeling and numerics for a gravity driven, zero rotation, thin coating flow on a horizontal cylinder; Schwartz and Weidner [36] consider this flow on a general curved surface. Evans, Schwartz, and Roy [22]

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present modeling and numerics for a gravity driven thin film flow on a rotating horizontal cylinder; Benjamin, Pritchard, and Tavener [4] present computation, modeling, and experiments for this flow inside a cylinder. For additional information about studies of thin liquid films, we refer readers to the excellent survey articles by Oron, Davis, and Bankoff [31] and by Craster and Matar [18].

Surprisingly little is understood about the initial value problem for (1.3). Bernis and Friedman [8] were the first to prove the existence of nonnegative weak solutions for nonnegative initial data for the related fourth-order nonlinear degenerate parabolic PDE

$$h_t + (f(h)\, h_{xxx})_x = 0, \tag{1.4}$$

where $f(h) = |h|^n\, f_0(h), f_0(h) > 0, n \geqslant 1$.

Unlike for second-order parabolic equations, there is no comparison principle for (1.4). For example, if the initial data is bounded below by 1, this does not ensure that the resulting solution will also be bounded below by 1. However, the degeneracy $f(h)$ in (1.4) is key in ensuring that, given nonnegative initial data, there is a nonnegative solution.

Lower-order terms can be added to (1.4) to model additional physical effects. For example,

$$h_t + (f(h)\, h_{xxx})_x - (g(h) h_x)_x = 0, \tag{1.5}$$

where $g(h) > 0$ for $h \neq 0$. Equation (1.5) can model a thin liquid film on a horizontal surface with gravity acting towards the surface. If this surface is not horizontal, then the dynamics can be modeled by

$$h_t + (h^n(a - b\, h_x + h_{xxx}))_x = 0, \quad a > 0, \quad b \geq 0. \tag{1.6}$$

The constant a in the first-order term vanishes as the surface becomes more and more horizontal. If the thin film of liquid is on a horizontal surface with gravity acting away from the surface, then the thin film dynamics can be modeled by

$$h_t + (f(h)\, h_{xxx})_x + (g(h) h_x)_x = 0. \tag{1.7}$$

In (1.5) and (1.6), the second-order term is stabilizing: if one linearizes the equation about a constant, positive steady state, then the presence of the second-order term increases how quickly perturbations decay in time. In (1.7), the second-order term is destabilizing: the linearized equation can have some long-wavelength perturbations that grow in time. For this reason, we refer to (1.7) as "long-wave unstable." The long-wave stable equations (1.5) and (1.6) have similar dynamics to those of (1.4); however, the long-wave unstable equation (1.7) can have nontrivial exact solutions and can have finite-time blowup ($h(x^*, t) \uparrow \infty$ as $t \uparrow t^* < \infty$).

In all cases, the fourth-order term makes it harder to prove desirable properties such as the short-time (or long-time) existence of nonnegative solutions given nonnegative initial data, compactly supported initial data yielding compactly supported solutions (finite speed of propagation), and uniqueness. Indeed, there are counterexamples to uniqueness of weak solutions [5]. Results about existence and long-time behavior for solutions of (1.5) can be found in [10]; analogous results for (1.6) are in [23]. See [12, 13] for results about existence, finite speed of propagation, and finite-time blowup for (1.7).

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In this paper we study the existence of weak solutions of the thin film equation

$$h_t + \big(|h|^3(a_0\, h_{xxx} + a_1 h_x + a_2 w'(x))\big)_x + a_3 h_x = 0, \tag{1.8}$$

where a_1, a_2, a_3 are arbitrary constants, constant $a_0 > 0$, and $w(x)$ is periodic. Equation (1.3) is a special case of (1.8). The sign of a_1 determines whether (1.8) is long-wave unstable. Also, the coefficient of the convection term $a_2(w'(x)|h|^3)_x$ can depend on space and will change sign if $a_2 w'(x) \not\equiv 0$. The cubic nonlinearity $|h|^3$ in (1.8) arises naturally in models of thin liquid films with no-slip boundary conditions at the liquid/solid interface. Our methods generalize naturally to $f(h) = |h|^n$; we refer the reader to [5, 8, 11] for the types of results expected.

Given nonnegative initial data that satisfies some reasonable conditions, we prove long-time existence of nonnegative periodic generalized weak solutions to the initial value problem for (1.8). We start by using energy methods to prove short-time existence of a weak solution and find an explicit lower bound on the time of existence. A generalization and sharpening of the method used in [12] allows us to prove that the H^1 norm of the constructed solution can grow at most linearly in time, precluding the possibility of a finite-time blowup. This H^1 control, combined with the explicit lower bound on the (short) time of existence, allows us to continue the weak solution in time, extending the short-time result to a long-time result.

If $a_2 = 0$ or $a_3 = 0$ in (1.8), then solutions will be uniformly bounded for all time. If $a_2 \neq 0$ and $a_3 \neq 0$, it is natural to ask if the nonlinear advection term could cause finite-time blowup ($h(x^*, t) \uparrow \infty$ as $t \uparrow t^*$). Such finite-time blowup is impossible by the linear-in-time bound on H^1, but we have not ruled out that a solution might grow in an unbounded manner as time goes to infinity.

In [14, 19], the authors consider the multidimensional analogue of (1.4),

$$h_t + \nabla \cdot (|h|^n \nabla \Delta h) = 0 \tag{1.9}$$

for $h(x,t)$, where $x \in \Omega \subset \mathbb{R}^N$ with $N = 2, 3$. Depending on the sign of A', if $g = 0$, then equation

$$h_t + \nabla \cdot (f(h)\nabla \Delta h + \nabla A(h)) = g(t, x, h, \nabla h) \tag{1.10}$$

on Ω is the multidimensional analogue of (1.5) or (1.7). In [20], the authors consider the long-wave stable case with $g = 0$ and power-law coefficients, $f(h) = |h|^n$ and $A'(h) = -|h|^m$. In [24], the author considers the Neumann problem for both the long-wave stable and unstable cases with the assumption that $f(h) \geq 0$ has power-law-like behavior near $h = 0$, that $|A'(h)|$ is dominated by $f(h)$ (specifically $|A'(h)| \leqslant d_0 f(h)$ for some d_0), and that the source/sink term $g(t, x, h)$ grows no faster than linearly in h. In [39, 40, 42], the authors consider the Neumann problem for the long-wave stable case of (1.10) with power-law coefficients and a larger class of source terms: $g(t, x, h) \sim |h|^{\lambda-1}h$ with $\lambda > 0$. In [37, 41], the same authors consider the long-wave stable equation with power-law coefficients but with $g(h) = \vec{a}\cdot\nabla b(h)$, where $b(z) \sim z^\lambda$ and $\vec{a} \in \mathbb{R}^N$: g models advective effects. They consider the problem both on $\mathbb{R}^N$ and on a bounded domain Ω.

All of these works on (1.9) and (1.10) construct nonnegative weak solutions from nonnegative initial data and address qualitative questions such as dependence on exponents n, m, and λ, on dimension N, speed of propagation of the support and of perturbations, exact asymptotics of the motion of the support, and positivity properties. We note that the works [37, 39, 40, 41, 42] also construct solutions with higher regularity properties ("strong" solutions).

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Finally, we refer readers to the technical report [17], which presents the results of this article, and some additional results, along with more extensive discussion, calculations, and simulations.

2. Steady state solutions. Smooth steady state solutions, $h(x,t) = h(x)$, of (1.3) satisfy

$$h - \tfrac{\mu}{3}h^3\,\cos(x) + \tfrac{\chi}{3}\,\left(h^3\,(h_x + h_{xxx})\right) = q, \tag{2.1}$$

where q is a constant of integration that corresponds to the dimensionless mass flux. In the zero surface tension case ($\chi = 0$), steady states satisfy

$$h - \tfrac{\mu}{3}h^3\,\cos(x) = q. \tag{2.2}$$

Such steady states were first studied by Johnson [25] and Moffatt [29]. Johnson proved that there are positive, unique, smooth steady states if and only if the flux is not too large: $0 < q < 2/(3\sqrt{\mu})$. These steady states are neutrally stable [30]. Smooth, positive steady states in the presence of surface tension have been studied by a number of authors. One striking computational result [2] is that for certain values of χ and μ there can be nonuniqueness.

These nonunique steady states were numerically discovered via an elegant combination of asymptotics and a two-parameter (mass and flux) continuation method [2, Figure 14]. To start the continuation method, earlier work [3] on the regime in which viscous forces dominate gravity was used. There, asymptotics show that for small fluxes the steady state is close to $q + 1/3q^3\cos(x) + \mathcal{O}(q^5)$, providing a good first guess for the iteration used to find the steady state. The bifurcation diagram shown in Figure 14 of [2] also suggests that the Moffatt model (1.2) can be considered as the limit of the Pukhnachov model (1.3) as surface tension goes to zero ($\chi \to 0$).

Pukhnachov proved [34] a nonexistence result: no positive steady states exist if $q > 2\sqrt{3/\mu} \simeq 3.464/\sqrt{\mu}$. We improve this, proving that no such solution exists if $q > 2/3\,\sqrt{2/\mu} \simeq 0.943/\sqrt{\mu}$.

PROPOSITION 2.1. *There does not exist a strictly positive 2π-periodic solution $h(x)$ of* (2.1) *if $q > 2/3\,\sqrt{2/\mu}$.*

Proof of Proposition 2.1. Following Pukhnachov, we start by rescaling the flux to 1 by introducing $y(x) = h(x)/q$ and introducing the parameters $\gamma = \frac{\chi q^3}{3}$ and $\beta = \frac{q^2\mu}{3}$. Equation (2.1) transforms into

$$\gamma(y''' + y') = \beta\cos(x) - \tfrac{1}{y^2} + \tfrac{1}{y^3}. \tag{2.3}$$

The solution y is written as $y(x) = a_0 + a_1\cos(x) + a_2\sin(x) + v(x)$, where $v(x) \perp \operatorname{span}\{1, \cos(x), \sin(x)\}$ and satisfies

$$\gamma(v''' + v') = \beta\cos(x) - \tfrac{1}{y(x)^2} + \tfrac{1}{y(x)^3}. \tag{2.4}$$

A solution v exists only if the right-hand side of (2.4) is orthogonal to $\operatorname{span}\{1, \cos(x), \sin(x)\}$. As a result,

$$\int_{-\pi}^{\pi}\left(\tfrac{1}{y(x)^2} - \tfrac{1}{y(x)^3}\right)\,dx = 0, \qquad \int_{-\pi}^{\pi}\left(\tfrac{1}{y(x)^2} - \tfrac{1}{y(x)^3}\right)\cos(x)\,dx = \pi\,\beta. \tag{2.5}$$

It follows from (2.5) that $\pi\beta \le \int_{y\ge 1}\frac{4}{27}\,(1+\cos(x))\,\,dx \le \frac{4}{27}\,2\pi$. This shows that if there is a positive steady state, then $\beta \le 8/27$. Recalling the definition of β, there is no steady state if $q > 2/3\,\sqrt{2/\mu}$. ∎

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The proof also holds in the case of zero surface tension $\chi = \gamma = 0$, and so it is natural that the bound $2/3\sqrt{2/\mu}$ is larger than $2/(3\sqrt{\mu})$ (the bound found by Johnson and Moffatt). Also, we note numerical simulations that suggest nonexistence of a positive steady state if $q > 0.854$ when $\mu = 1$ for a large range of surface tension values [26, p. 61]; our bound of 0.943 is not too far off from this.

3. Short-time existence and regularity of solutions. We are interested in the existence of nonnegative generalized weak solutions to the following initial-boundary value problem:

$$\text{(P)}\begin{cases} h_t + (f(h)(a_0 h_{xxx} + a_1 h_x + a_2 w'(x)))_x + a_3 h_x = 0 \text{ in } Q_T, & (3.1)\\ \frac{\partial^i h}{\partial x^i}(-a,t) = \frac{\partial^i h}{\partial x^i}(a,t) \text{ for } t > 0,\ i = \overline{0,3}, & (3.2)\\ h(x,0) = h_0(x) \geqslant 0, & (3.3)\end{cases}$$

where $f(h) = |h|^3$, $h = h(x,t)$, $\Omega = (-a,a)$, and $Q_T = \Omega \times (0,T)$. Note that rather than considering the interval $(-a,a)$ with boundary conditions (3.2), one can equally as well consider the problem on the circle S^1; our methods and results would apply here too. Recall that a_1, a_2, and a_3 in (3.1) are arbitrary constants; a_0 is required to be positive. The function w in (3.1) is assumed to satisfy

$$w \in C^{2+\gamma}(\Omega) \text{ for some } 0 < \gamma < 1, \quad \frac{\partial^i w}{\partial x^i}(-a) = \frac{\partial^i w}{\partial x^i}(a) \text{ for } i = \overline{0,2}. \tag{3.4}$$

We consider a generalized weak solution in the following sense [5, 6].

DEFINITION 3.1. *A generalized weak solution of problem* (P) *is a function* h *satisfying*

$$h \in C^{1/2,1/8}_{x,t}(\overline{Q}_T) \cap L^\infty(0,T;H^1(\Omega)), \tag{3.5}$$

$$h_t \in L^2(0,T;(H^1(\Omega))'), \tag{3.6}$$

$$h \in C^{4,1}_{x,t}(\mathcal{P}),\quad \sqrt{f(h)}\,(a_0 h_{xxx} + a_1 h_x + a_2 w') \in L^2(\mathcal{P}), \tag{3.7}$$

where $\mathcal{P} = \overline{Q}_T \setminus (\{h = 0\} \cup \{t = 0\})$ *and* h *satisfies* (3.1) *in the following sense:*

$$\int_0^T \langle h_t(\cdot,t), \phi\rangle\, dt - \iint_{\mathcal{P}} f(h)(a_0 h_{xxx} + a_1 h_x + a_2 w'(x))\phi_x\, dxdt \\ - a_3 \iint_{Q_T} h\phi_x\, dxdt = 0 \tag{3.8}$$

for all $\phi \in C^1(Q_T)$ *with* $\phi(-a,\cdot) = \phi(a,\cdot)$;

$$h(\cdot,t) \to h(\cdot,0) = h_0 \textit{ pointwise and strongly in } L^2(\Omega) \textit{ as } t \to 0, \tag{3.9}$$

$$h(-a,t) = h(a,t)\ \forall t \in [0,T] \textit{ and } \frac{\partial^i h}{\partial x^i}(-a,t) = \frac{\partial^i h}{\partial x^i}(a,t) \\ \textit{for } i = \overline{1,3} \textit{ at all points of the lateral boundary where } \{h \neq 0\}. \tag{3.10}$$

Because the second term of (3.8) has an integral over $\mathcal{P}$ rather than over Q_T, the generalized weak solution is "weaker" than a standard weak solution. Also note that the first term of (3.8) uses $h_t \in L^2(0,T;(H^1(\Omega))')$; this is different from the definition of weak solution first introduced by Bernis and Friedman [8]; there, the first term was the integral of $h\phi_t$ integrated over Q_T.

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We first prove the short-time existence of a generalized weak solution and then prove that it can have additional regularity. In section 4 we prove additional control for the H^1 norm which then allows us to prove long-time existence.

THEOREM 1 (existence). *Let the nonnegative initial data $h_0 \in H^1(\Omega)$ satisfy*

$$\int_\Omega \tfrac{1}{h_0(x)}\,dx < \infty, \tag{3.11}$$

and either (1) $h_0(-a) = h_0(a) = 0$ *or* (2) $h_0(-a) = h_0(a) \neq 0$ *and* $\frac{\partial^i h_0}{\partial x^i}(-a) = \frac{\partial^i h_0}{\partial x^i}(a)$ *holds for* $i = \overline{1,3}$. *Then for some time $T_{loc} > 0$ there exists a nonnegative generalized weak solution, h, on $Q_{T_{loc}}$ in the sense of Definition* 3.1. *Furthermore,*

$$h \in L^2(0, T_{loc}; H^2(\Omega)). \tag{3.12}$$

Let

$$\mathcal{E}_0(T) := \tfrac{1}{2}\int_\Omega (a_0 h_x^2(x,T) - a_1 h^2(x,T) - 2a_2 w(x) h(x,T))\,dx. \tag{3.13}$$

Then the weak solution satisfies

$$\mathcal{E}_0(T_{loc}) + \iint\limits_{\{h>0\}} h^3(a_0 h_{xxx} + a_1 h_x + a_2 w')^2\,dx\,dt \leqslant \mathcal{E}_0(0) + K\,T_{loc}, \tag{3.14}$$

where $K = |a_2 a_3|\,\|w'\|_\infty M < \infty$. The time of existence, T_{loc}, is determined by a_0, a_1, a_2, w', $|\Omega|$, and h_0.

We note that the analogue of Theorem 4.2 in [8] also holds: there exists a nonnegative weak solution with the integral formulation

$$\begin{aligned} \int_0^T \langle h_t(\cdot,t), \phi\rangle\,dt + a_0 \iint\limits_{Q_T} (3h^2 h_x h_{xx}\phi_x + h^3 h_{xx}\phi_{xx})\,dxdt \\ - \iint\limits_{Q_T} \left(a_1 h^3 h_x + a_2 h^3 w' + a_3 h\right)\phi_x\,dxdt = 0. \end{aligned} \tag{3.15}$$

THEOREM 2 (regularity). *If the initial data from Theorem* 1 *also satisfies*

$$\int_\Omega h_0^{\alpha-1}(x)\,dx < \infty \tag{3.16}$$

for some $-1/2 < \alpha < 1$, $\alpha \neq 0$, then there exists $0 < T_{loc}^{(\alpha)} \leq T_{loc}$ such that there exists a nonnegative generalized weak solution that satisfies Theorem 1 *and has the extra regularity*

$$h^{\frac{\alpha+2}{2}} \in L^2(0, T_{loc}^{(\alpha)}; H^2(\Omega)) \quad \text{and} \quad h^{\frac{\alpha+2}{4}} \in L^2(0, T_{loc}^{(\alpha)}; W_4^1(\Omega)).$$

The solutions from Theorem 2 are often called "strong" solutions in the thin film literature.

If the initial data satisfy (3.16), then the added regularity from Theorem 2 allows one to prove the existence of nonnegative solutions with an integral formulation [11]

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that is similar to that of (3.15), except that the second integral is replaced by the results of one more integration by parts (there are no h_{xx} terms).

If one considers problem (P) with nonlinearity $f(h) = |h|^n$, then for $n \in (0, 3)$, Theorems 1 and 2 would hold for general nonnegative initial data $h_0 \in H^1(\Omega)$. If $n > 3$, then these theorems would also hold if the initial data satisfy the analogues of conditions (3.11) and (3.16), $\int h_0^{2-n}\, dx < \infty$ and $\int h_0^{\alpha+2-n}\, dx < \infty$, respectively. We refer the reader to [11, 5] for the techniques that would be needed to generalize Theorems 1 and 2 in this way.

The proofs of Theorems 1 and 2 rely on approximate solutions and a priori control of their "energy" and "entropy" at all moments in time. The energy at time T is the first-order[1] functional (3.13). Similarly, the entropies at time T are the zeroth-order functionals (3.11) and (3.16) evaluated for $h(\cdot, T)$.

Bernis and Friedman [8] were the first to introduce this energy–entropy approach for thin film equations; they proved the existence of generalized weak solutions for $h_t = -(h^n h_{xxx})_x$ using the energy $\int h_x^2$ and the entropy $\int h^{2-n}$. Since then, there has been great development in energy–entropy methods.

For example, Bontat et al. [16] consider

$$h_t = -\left(h^n h_{xxx} + \alpha h^{n-1} h_x h_{xx} + \beta h^{n-2} (h_x)^3\right)_x$$

as well its analogue in two and three space dimensions.

Rakotoson, Rakotoson, and Verbeke [35] consider

$$h_t = \beta h_{xxxxxx} + \left(h h_{xx} + (1/2 - \alpha)(h_x)^2\right)_{xx} - \gamma \left(\frac{(h_x)^3}{h}\right)_x$$

for $\beta \geq 0$.

Both works use the energy $\int h_x^2$ and an entropy $\int G(h)$. Unlike Bernis and Friedman's entropy, the function $G(y)$ has a piecewise definition for $y > 0$ and $y < 0$.

In [43], Ulusoy considers

$$h_t = -\left(h^n \left((p-1)(h_x^2)^{p/2-1} h_{xx}\right)_x\right)_x,$$

which has a gradient flow structure based on the energy $\int |h_x|^p$. Using this and the entropy $\int h^{2-n}$, the author proves the existence of nonnegative weak solutions for $p \neq 2$.

Ansini and Giacomelli [1] consider the doubly nonlinear thin film equation

$$h_t = -\left(|h|^n |h_{xxx}|^{p-2} h_{xxx}\right)_x$$

with $p \geq 2$ and $n \in (0, p+1)$. For $p > 2$, they do not have an entropy to work with. Instead, they make subtle use of the energy $\int h_x^2$ and the rate of energy dissipation aided by Bernis' interpolation inequalities [7].

3.1. Regularized problem. Given $\delta, \varepsilon > 0$, a regularized parabolic problem, similar to that of Bernis and Friedman [8], is considered:

$$(\mathrm{P}_{\delta,\varepsilon}) \begin{cases} h_t + \left(f_{\delta\varepsilon}(h)\left(a_0 h_{xxx} + a_1 h_x + a_2 w'(x)\right)\right)_x + a_3 h_x = 0, & (3.17) \\ \frac{\partial^i h}{\partial x^i}(-a, t) = \frac{\partial^i h}{\partial x^i}(a, t) \text{ for } t > 0,\ i = \overline{0, 3}, & (3.18) \\ h(x, 0) = h_{0,\varepsilon}(x), & (3.19) \end{cases}$$

[1] We call the functional first-order because the integrand depends on the first derivative of h.

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where

$$f_{\delta\varepsilon}(z) := f_\varepsilon(z) + \delta = \tfrac{|z|^4}{|z|+\varepsilon} + \delta \quad \forall\, z \in \mathbb{R}^1,\ \delta > 0,\ \varepsilon > 0. \tag{3.20}$$

The $\delta > 0$ in (3.20) makes the problem (3.17) regular (i.e., uniformly parabolic). The parameter ε is an approximating parameter which has the effect of increasing the degeneracy from $f(h) \sim |h|^3$ to $f_\varepsilon(h) \sim h^4$. The nonnegative initial data, h_0, is approximated via

$$\begin{gathered} h_0 + \varepsilon^\theta \le h_{0,\varepsilon} \in C^{4+\gamma}(\overline{\Omega}) \text{ for some } \theta \in (0, 2/5) \text{ and } \gamma \text{ from (3.4)}, \\ \tfrac{\partial^i h_{0,\varepsilon}}{\partial x^i}(-a) = \tfrac{\partial^i h_{0,\varepsilon}}{\partial x^i}(a) \text{ for } i = \overline{0,3}, \\ h_{0,\varepsilon} \to h_0 \text{ strongly in } H^1(\Omega) \text{ as } \varepsilon \to 0. \end{gathered} \tag{3.21}$$

The role of ε in (3.21) is to smooth the initial data from $H^1(\Omega)$ to $C^{4+\gamma}(\overline{\Omega})$ and to "lift" the initial data, making it positive.

By Èĭdel'man [21, Theorem 6.3, p. 302], the regularized problem has a unique classical solution $h_{\delta\varepsilon} \in C^{4+\gamma,1+\gamma/4}_{x,t}(\Omega \times [0, \tau_{\delta\varepsilon}])$ for some time $\tau_{\delta\varepsilon} > 0$. For any fixed value of δ and ε, by Èĭdel'man [21, Theorem 9.3, p. 316] if one can prove a uniform in time a priori bound $|h_{\delta\varepsilon}(x,t)| \le A_{\delta\varepsilon} < \infty$ for some longer time interval $[0, T_{loc,\delta\varepsilon}]$ ($T_{loc,\delta\varepsilon} > \tau_{\delta\varepsilon}$) and for all $x \in \Omega$, then Schauder-type interior estimates [21, Corollary 2, p. 213] imply that the solution $h_{\delta\varepsilon}$ can be continued in time to be in $C^{4+\gamma,1+\gamma/4}_{x,t}(\Omega \times [0, T_{loc,\delta\varepsilon}])$.

Although the solution $h_{\delta\varepsilon}$ is initially positive, there is no guarantee that it will remain nonnegative. The goal is to take $\delta \to 0$, $\varepsilon \to 0$ in such a way that (1) $T_{loc,\delta\varepsilon} \to T_{loc} > 0$, (2) the solutions $h_{\delta\varepsilon}$ converge to a (nonnegative) limit, h, which is a generalized weak solution, and (3) h inherits certain a priori bounds. This is done by proving various a priori estimates for $h_{\delta\varepsilon}$ that are uniform in δ and ε and hold on a time interval $[0, T_{loc}]$ that is independent of δ and ε. As a result, $\{h_{\delta\varepsilon}\}$ will be a uniformly bounded and equicontinuous (in the $C^{1/2,1/8}_{x,t}$ norm) family of functions in $\bar{\Omega} \times [0, T_{loc}]$. Taking $\delta \to 0$ will result in a family of functions $\{h_\varepsilon\}$ that are classical, positive, unique solutions to the regularized problem with $\delta = 0$. Taking $\varepsilon \to 0$ will then result in the desired generalized weak solution h. This last step is where the possibility of nonunique weak solutions arise; see [5] for simple examples of how such constructions applied to $h_t = -(|h|^n h_{xxx})_x$ can result in two different solutions arising from the same initial data.

3.2. A priori estimates. Our first task is to derive a priori estimates for classical solutions of (3.17)–(3.21). The lemmas in this section are proved in Appendix A.

We use an integral quantity based on a function $G_{\delta\varepsilon}$ chosen so that

$$G''_{\delta\varepsilon}(z) = \tfrac{1}{f_{\delta\varepsilon}(z)} \quad \text{and} \quad G_{\delta\varepsilon}(z) \ge 0. \tag{3.22}$$

This is analogous to the "entropy" function first introduced by Bernis and Friedman [8].

Lemma 3.1. *There exists $\delta_0 > 0$, $\varepsilon_0 > 0$, and time $T_{loc} > 0$ such that if $\delta \in [0, \delta_0)$, $\varepsilon \in (0, \varepsilon_0)$, if $h_{\delta\varepsilon}$ is a classical solution of the problem* (3.17)–(3.21) *with initial data $h_{0,\varepsilon}$, and if $h_{0,\varepsilon}$ satisfies* (3.21) *and is built from a nonnegative function h_0 that*

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satisfies the hypotheses of Theorem 1, *then for any* $T \in [0, T_{loc}]$ *the solution* $h_{\delta\varepsilon}$ *satisfies*

$$\int_\Omega \left\{ h^2_{\delta\varepsilon,x}(x,T) + \tfrac{|a_1|}{a_0}\left(\tfrac{|a_1|}{a_0} + 2\delta\right) G_{\delta\varepsilon}(h_{\delta\varepsilon}(x,T)) \right\} dx + a_0 \iint\limits_{Q_T} f_{\delta\varepsilon}(h_{\delta\varepsilon}) h^2_{\delta\varepsilon,xxx}\, dxdt \leqslant K_1 < \infty, \tag{3.23}$$

$$\int_\Omega G_{\delta\varepsilon}(h_{\delta\varepsilon}(x,T))\, dx + a_0 \iint\limits_{Q_T} h^2_{\delta\varepsilon,xx}\, dxdt \le K_2 < \infty, \tag{3.24}$$

and the energy $\mathcal{E}_{\delta\varepsilon}(t)$ *(see* (3.13)*) satisfies*

$$\mathcal{E}_{\delta\varepsilon}(T) + \iint\limits_{Q_T} f_{\delta\varepsilon}(h_{\delta\varepsilon})(a_0 h_{\delta\varepsilon,xxx} + a_1 h_{\delta\varepsilon,x} + a_2 w')^2\, dxdt \leqslant C_0 + K_3 T, \tag{3.25}$$

where $K_3 = |a_2 a_3|\, \| w' \|_\infty M < \infty$. *The time* T_{loc} *and the constants* K_1, K_2, C_0, *and* K_3 *are independent of* δ *and* ε.

The existence of δ_0, ε_0, T_{loc}, K_1, K_2, and K_3 is constructive; how to find them and what quantities determine them is shown in Appendix A.

Lemma 3.1 yields uniform-in-δ-and-ε bounds for $\int h^2_{\delta\varepsilon,x}$, $\int G_{\delta\varepsilon}(h_{\delta\varepsilon})$, $\iint h^2_{\delta\varepsilon,xx}$, and $\iint f_{\delta\varepsilon}(h_{\delta\varepsilon}) h^2_{\delta\varepsilon,xxx}$. However, these bounds are found in a different manner than in earlier work for the equation $h_t = -(|h|^n h_{xxx})_x$, for example. Although inequality (3.24) is unchanged, inequality (3.23) has an extra term involving $G_{\delta\varepsilon}$. In the proof, this term was introduced to control additional, lower-order terms. This idea of a "blended" $\|h_x\|_2$-entropy bound was first introduced by Shishkov and Taranets especially for long-wave stable thin film equations with convection [37].

The final a priori bound uses the following functions, parametrized by α,

$$G^{(\alpha)}_\varepsilon(z) := \tfrac{z^{\alpha-1}}{(\alpha-1)(\alpha-2)} + \tfrac{\varepsilon z^{\alpha-2}}{(\alpha-3)(\alpha-2)};\ (G^{(\alpha)}_\varepsilon(z))'' = \tfrac{z^\alpha}{f_\varepsilon(z)}. \tag{3.26}$$

LEMMA 3.2. *Assume* ε_0 *and* T_{loc} *are from Lemma* 3.1, $\delta = 0$, *and* $\varepsilon \in (0, \varepsilon_0)$. *Assume* h_ε *is a positive, classical solution of the problem* (3.17)–(3.21) *with initial data* $h_{0,\varepsilon}$ *satisfying Lemma* 3.1. *Fix* $\alpha \in (-1/2, 1)$ *with* $\alpha \neq 0$. *If the initial data* $h_{0,\varepsilon}$ *is built from* h_0 *which also satisfies*

$$\int_\Omega h_0^{\alpha-1}(x)\, dx < \infty, \tag{3.27}$$

then there exists $\varepsilon_0^{(\alpha)}$ *and* $T^{(\alpha)}_{loc}$ *with* $0 < \varepsilon_0^{(\alpha)} \le \varepsilon_0$ *and* $0 < T^{(\alpha)}_{loc} \le T_{loc}$ *such that*

$$\int_\Omega \{ h^2_{\varepsilon,x}(x,T) + G^{(\alpha)}_\varepsilon(h_\varepsilon(x,T)) \}\, dx + \iint\limits_{Q_T} \left[\beta h_\varepsilon^\alpha h^2_{\varepsilon,xx} + \gamma h_\varepsilon^{\alpha-2} h^4_{\varepsilon,x} \right] dx\, dt \leqslant K_4 < \infty \tag{3.28}$$

holds for all $T \in [0, T^{(\alpha)}_{loc}]$ *and some constant* K_4 *that is determined by* α, ε_0, a_0, a_1, a_2, w', Ω, *and* h_0. *Here,*

$$\beta = \begin{cases} a_0 & \text{if } \alpha \in (0,1), \\ a_0 \frac{1+2\alpha}{4(1-\alpha)} & \text{if } \alpha \in (-1/2, 0), \end{cases} \quad \gamma = \begin{cases} a_0 \frac{\alpha(1-\alpha)}{6} & \text{if } \alpha \in (0,1), \\ a_0 \frac{(1+2\alpha)(1-\alpha)}{36} & \text{if } \alpha \in (-1/2, 0). \end{cases}$$

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Furthermore,

$$h_\varepsilon^{\frac{\alpha+2}{2}} \in L^2(0,T_{loc};H^2(\Omega)) \quad \textit{and} \quad h_\varepsilon^{\frac{\alpha+2}{4}} \in L^2(0,T_{loc};W_4^1(\Omega)) \tag{3.29}$$

with a uniform-in-ε bound.

The α-entropy, $\int G_0^{(\alpha)}(h)\,dx$, was first introduced for $\alpha=-1/2$ in [9], and an a priori bound like that of Lemma 3.2 and regularity results like those of Theorem 2 were found simultaneously and independently in [5] and [11].

3.3. Proof of existence and regularity of solutions. Bound (3.23) yields uniform L^∞ control for classical solutions $h_{\delta\varepsilon}$, allowing the time of existence $T_{loc,\delta\varepsilon}$ to be taken as T_{loc} for all $\delta \in (0,\delta_0)$ and $\varepsilon \in (0,\varepsilon_0)$. The existence theory starts by constructing a classical solution $h_{\delta\varepsilon}$ on $[0,T_{loc}]$ that satisfies the hypotheses of Lemma 3.1 if $\delta \in (0,\delta_0)$ and $\varepsilon \in (0,\varepsilon_0)$. The regularizing parameter, δ, is taken to zero, and one proves that there is a limit h_ε and that h_ε is a generalized weak solution. One then proves additional regularity for h_ε, specifically that it is strictly positive, classical, and unique. It then follows that the a priori bounds given by Lemmas 3.1 and 3.2 apply to h_ε. This allows us to take the approximating parameter, ε, to zero and construct the desired generalized weak solution of Theorems 1 and 2.

LEMMA 3.3. *Assume that the initial data $h_{0,\varepsilon}$ satisfies* (3.21) *and is built from a nonnegative function h_0 that satisfies the hypotheses of Theorem* 1. *Fix $\delta=0$ and $\varepsilon\in(0,\varepsilon_0)$, where ε_0 is from Lemma* 3.1. *Then there exists a unique, positive, classical solution h_ε on $[0,T_{loc}]$ of problem $(P_{0,\varepsilon})$ (see* (3.17)–(3.21)*) with initial data $h_{0,\varepsilon}$, where T_{loc} is the time from Lemma* 3.1.

Proof. Arguing the same way as Bernis and Friedman [8] one can construct a generalized weak solution h_ε. We now prove that this h_ε is a strictly positive, classical, unique solution. This uses the entropy $\int G_{\delta\varepsilon}(h_{\delta\varepsilon})$ and the a priori bound (3.24). This bound is, up to the coefficient a_0, identical to the a priori bound (4.17) in [8]. By construction, the initial data $h_{0,\varepsilon}$ is positive (see (3.21)); hence $\int G_\varepsilon(h_{0,\varepsilon})\,dx<\infty$. Also, by construction $f_\varepsilon(z)\sim z^4$ for $z \ll 1$. This implies that the generalized weak solution h_ε is strictly positive [8, Theorem 4.1]. Because the initial data $h_{0,\varepsilon}$ is in $C^{4+\gamma}(\overline{\Omega})$, it follows that h_ε is a classical solution in $C^{4,1}_{x,t}(\overline{Q_{T_{loc}}})$. The proof of Theorem 4.1 in [8] then implies that h_ε is unique. ☐

Proof of Theorem 1. As in the proof of Lemma 3.3, following [8], there is a subsequence $\{\varepsilon_k\}$ such that h_{ε_k} converges uniformly to a function $h \in C_{x,t}^{1/2,1/8}$ which is a generalized weak solution in the sense of Definition 3.1 with $f(h)=|h|^3$.

The initial data are assumed to have finite entropy: $\int 1/h_0<\infty$. This, combined with $f(h)=|h|^3$, implies that the generalized weak solution h is nonnegative and the set of points $\{h=0\}$ in $Q_{T_{loc}}$ has zero measure [8, Theorem 4.1].

To prove (3.14), start by taking $T=T_{loc}$ in the a priori bound (3.25). As $\varepsilon_k\to0$, the right-hand side of (3.25) is unchanged. First, consider the $\varepsilon_k\to0$ limit of

$$\mathcal{E}_{\varepsilon_k}(T_{loc}) = \tfrac{1}{2}\int_\Omega a_0 h^2_{\varepsilon_k,x}(x,T_{loc}) - a_1 h^2_{\varepsilon_k}(x,T_{loc}) - 2a_2 w(x) h_{\varepsilon_k}(x,T_{loc})dx.$$

By the uniform convergence of h_{ε_k} to h, the second and third terms in the energy converge strongly as $\varepsilon_k\to0$. The bound (3.25) yields a uniform bound on $\{\int_\Omega h^2_{\varepsilon_k,x}(x,T_{loc})\,dx\}$. Taking a further refinement of $\{\varepsilon_k\}$ yields $h_{\varepsilon_k,x}(\cdot,T_{loc})$ converging weakly in $L^2(\Omega)$. In a Hilbert space, the norm of the weak limit is less than or equal to the lim inf of the norms of the functions in the sequence; hence

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$\mathcal{E}_0(T_{loc}) \leq \liminf_{\varepsilon_k \to 0} \mathcal{E}_{\varepsilon_k}(T_{loc})$. A uniform bound on $\iint f_\varepsilon(h_\varepsilon)\left(a_0 h_{\varepsilon,xxx} + \cdots\right)^2\, dx$ also follows from (3.25). Hence $\sqrt{f_{\varepsilon_k}(h_{\varepsilon_k})}\,(a_0 h_{\varepsilon_k,xxx} + \cdots)$ converges weakly in $L^2(Q_{T_{loc}})$, after taking a further subsequence. It suffices to determine the weak limit up to a set of measure zero. Because $h \geq 0$ and $\{h = 0\}$ has measure zero, it suffices to determine the weak limit on $\{h > 0\}$.

The regularity theory for parabolic equations allows one to argue that $h \in C^{4,1}_{x,t}(\mathcal{P})$, and the weak limit is $h^{3/2}\,(a_0 h_{xxx} + \cdots)$ on $\{h > 0\}$. Using that (1) the norm of the weak limit is less than or equal to the lim inf of the norms of the functions in the sequence and that (2) the lim inf of a sum is greater than or equal to the sum of the lim infs results in the desired bound (3.14).

It follows from (3.24) that $h_{\varepsilon_k,xx}$ converges weakly to some v in $L^2(Q_{T_{loc}})$, combining with strong convergence in $L^2(0,T;H^1(\Omega))$ of h_{ε_k} to h by Lemma B.1, and with the definition of weak derivative, we obtain that $v = h_{xx}$ and $h \in L^2(0,T_{loc}; H^2(\Omega))$, which implies (3.12). Hence $h_{\varepsilon,t} \to h_t$ weakly in $L^2(0,T;(H^1(\Omega))')$, which implies (3.6). By Lemma B.2 we also have $h \in C([0,T_{loc}],L^2(\Omega))$. $\square$

Proof of Theorem 2. Fix $\alpha \in (-1/2, 1)$. The initial data h_0 is assumed to have finite entropy $\int G_0^{(\alpha)}(h_0(x))\, dx < \infty$; hence Lemma 3.2 holds for the approximate solutions $\{h_{\varepsilon_k}\}$, where this sequence of approximate solutions is assumed to be the one at the end of the proof of Theorem 1. By (3.29),

$$\left\{h_{\varepsilon_k}^{\frac{\alpha+2}{2}}\right\} \quad \text{is uniformly bounded in } \varepsilon_k \text{ in } L^2(0,T_{loc};H^2(\Omega))$$

and

$$\left\{h_{\varepsilon_k}^{\frac{\alpha+2}{4}}\right\} \quad \text{is uniformly bounded in } \varepsilon_k \text{ in } L^2(0,T_{loc};W_4^1(\Omega)).$$

Taking a further subsequence in $\{\varepsilon_k\}$, it follows from the proof of [19, Lemma 2.5, p. 330] that these sequences converge weakly in $L^2(0,T_{loc};\ H^2(\Omega))$ and $L^2(0,T_{loc}; W_4^1(\Omega))$ to $h^{\frac{\alpha+2}{2}}$ and $h^{\frac{\alpha+2}{4}}$, respectively. $\square$

4. Long-time existence of solutions.

LEMMA 4.1. *Let $h \in H^1(\Omega)$ be a nonnegative function and $\int_\Omega h(x)\,dx = M$. Then*

$$\|h\|^2_{L^2(\Omega)} \leqslant 6^{\frac{2}{3}} M^{\frac{4}{3}} \left(\int_\Omega h_x^2\, dx\right)^{\frac{1}{3}} + \tfrac{M^2}{|\Omega|}. \tag{4.1}$$

Note that by taking h to be a constant function, one finds that the constant $M^2/|\Omega|$ in (4.1) is sharp.

Proof. Let $v = h - M/|\Omega|$. By (A.3),

$$\|v\|^2_{L^2(\Omega)} \leqslant (\tfrac{3}{2})^{\frac{2}{3}} \left(\int_\Omega v_x^2\, dx\right)^{\frac{1}{3}} \left(\int_\Omega |v|\, dx\right)^{\frac{4}{3}}.$$

Hence,

$$\begin{aligned}\|h\|^2_{L^2(\Omega)} &\leqslant (\tfrac{3}{2})^{\frac{2}{3}} \left(\int_\Omega h_x^2\, dx\right)^{\frac{1}{3}} \left(\int_\Omega \left|h - \tfrac{M}{|\Omega|}\right|\, dx\right)^{\frac{4}{3}} + \tfrac{M^2}{|\Omega|}\\ &\leqslant (\tfrac{3}{2})^{\frac{2}{3}} \left(\int_\Omega h_x^2\, dx\right)^{\frac{1}{3}} (2M)^{\frac{4}{3}} + \tfrac{M^2}{|\Omega|}. \qquad \square\end{aligned}$$

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Lemma 4.1 and the bound (3.14) are used to prove H^1 control of the generalized weak solution constructed in Theorem 1.

LEMMA 4.2. *Let h be the generalized solution of Theorem* 1. *Then*

$$\frac{a_0}{4}\,\|h(\cdot,T_{loc})\|^2_{H^1(\Omega)} \le \mathcal{E}_0(0) + KT_{loc} + K_3, \tag{4.2}$$

where $\mathcal{E}_0(0)$ is defined in (3.13), $M = \int h_0$, $K = |a_2 a_3|\|w'\|_\infty M$, *and*

$$K_3 = \begin{cases} |a_2|\|w\|_\infty M & \text{if } a_0 + a_1 \le 0, \\ |a_2|\|w\|_\infty M + M^2\left(\frac{2\sqrt{6}\,(a_0+a_1)^{3/2}}{3\sqrt{a_0}} + \frac{a_0+a_1}{2|\Omega|}\right) & \text{otherwise.} \end{cases}$$

Note that if the evolution is missing either linear or nonlinear advection ($a_2 = 0$ or $w' = 0$ or $a_3 = 0$), then Lemma 4.2 provides a uniform-in-time upper bound for $\|h(\cdot,T_{loc})\|_{H^1}$.

For (1.3), which models the flow of a thin film of liquid on the outside of a rotating cylinder, one has $a_0 = a_1 = \frac{\chi}{3}$, $a_2 = -\frac{\mu}{3}$, $a_3 = 1$, $w(x) = \sin x$, and $|\Omega| = 2\pi$. In this case, the H^1 bound (4.2) becomes

$$\frac{\chi}{12}\|h(\cdot,T_{loc})\|^2_{H^1(\Omega)} \leqslant \mathcal{E}_0(0) + \frac{\mu}{3}CT_{loc} + \frac{\mu}{3}M + M^2\left(\frac{8}{3}\sqrt{\chi} + \frac{\chi}{6\pi}\right),$$

where $2\mathcal{E}_0(0) = \int(\chi/3\,(h^2_{0,x} - h_0^2) + 2\mu/3\,\sin(x)\,h_0\,)dx$. The H^1 bound (4.2) actually holds true for all times for which h is strictly positive. Recalling the definition (1.1) of χ, one sees that the H^1 control is lost as $\chi \to 0$ (i.e., as $\sigma/(\nu\rho R\omega) \to 0$), for example, in the zero surface tension limit.

Proof. By (3.13),

$$\frac{a_0}{2}\int_\Omega h_x^2(x,T)\,dx = \mathcal{E}_0(T) + \frac{a_1}{2}\int_\Omega h^2(x,T)\,dx + a_2\int_\Omega h(x,T)\,w(x)\,dx.$$

The linear-in-time bound (3.14) on $\mathcal{E}_0(T_{loc})$ then implies

$$\frac{a_0}{2}\|h(\cdot,T_{loc})\|^2_{H^1} \le \mathcal{E}_0(0) + K\,T_{loc} + \frac{a_0+a_1}{2}\int_\Omega h^2\,dx + |a_2|\|w\|_\infty M \tag{4.3}$$

with $K = |a_2 a_3|\|w'\|_\infty M$.

Case 1. $a_0 + a_1 \le 0$. The third term on the right-hand side of (4.3) is nonpositive and can be removed. The desired bound (4.2) follows immediately.

Case 2. $a_0 + a_1 > 0$. By Lemma 4.1 and Young's inequality

$$\begin{aligned} \frac{a_0+a_1}{2}\int_\Omega h^2\,dx &\le \frac{a_0+a_1}{2}\left(6^{\frac{2}{3}}M^{\frac{4}{3}}\left(\int_\Omega h_x^2\,dx\right)^{\frac{1}{3}} + \frac{M^2}{|\Omega|}\right) \\ &\le \frac{a_0}{4}\int_\Omega h_x^2(x,T_{loc})\,dx + M^2\left(\frac{2\sqrt{6}(a_0+a_1)^{3/2}}{3\sqrt{a_0}} + \frac{a_0+a_1}{2|\Omega|}\right). \end{aligned} \tag{4.4}$$

Using this in (4.3), the desired bound (4.2) follows immediately. ∎

This H^1 control in time of the generalized solution is now used to extend the short-time existence result of Theorem 1 to a long-time existence result.

THEOREM 3. *Let T_g be an arbitrary positive finite number. The generalized weak solution h of Theorem* 1 *can be continued in time from $[0,T_{loc}]$ to $[0,T_g]$ in such a way that h is also a generalized weak solution and satisfies all the bounds of Theorem* 1 *(with T_{loc} replaced by T_g).*

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Similarly, the short-time existence of strong solutions (see Theorem 2) can be extended to a long-time existence.

Proof. To construct a weak solution up to time T_g, one applies the local existence theory iteratively, taking the solution at the final time of the current time interval as initial data for the next time interval.

Introduce the times

$$0 = T_0 < T_1 < T_2 < \cdots < T_N < \cdots, \quad \text{where} \quad T_N := \sum_{n=0}^{N-1} T_{n,loc} \tag{4.5}$$

and $T_{n,loc}$ is the interval of existence (A.12) for a solution with initial data $h(\cdot, T_n)$:

$$T_{n,loc} := \tfrac{9}{40c_6} \min\left\{1, \left(\int_\Omega h_x^2(x,T_n) + 2\tfrac{c_3}{a_0} G_0(h(x,T_n))\, dx\right)^{-2}\right\}. \tag{4.6}$$

The proof proceeds by contradiction. Assume there exists initial data h_0 satisfying the hypotheses of Theorem 1, which results in a weak solution that cannot be extended arbitrarily in time:

$$\sum_{k=0}^{\infty} T_{n,loc} = T^* < \infty \quad \Longrightarrow \quad \lim_{n\to\infty} T_{n,loc} = 0.$$

From the definition (4.6) of $T_{n,loc}$, this implies

$$\lim_{n\to\infty} \int_\Omega (h_x^2(x,T_n) + 2\tfrac{c_3}{a_0} G_0(h(x,T_n)))\, dx = \infty. \tag{4.7}$$

By (4.2) and (3.14),

$$\tfrac{a_0}{4} \int_\Omega h_x^2(x,T_n)\, dx \le \mathcal{E}_0(T_{n-1}) + K\, T_{n-1,loc} + K_3,$$

$$\mathcal{E}_0(T_{n-1}) \le \mathcal{E}_0(T_{n-2}) + K\, T_{n-2,loc}.$$

Combining these,

$$\tfrac{a_0}{4} \int_\Omega h_x^2(x,T_n)\, dx \le \mathcal{E}_0(T_{n-2}) + K\,(T_{n-2,loc} + T_{n-1,loc}) + K_3.$$

Continuing in this way,

$$\tfrac{a_0}{4} \int_\Omega h_x^2(x,T_n)\, dx \le \mathcal{E}_0(0) + K\, T_n + K_3. \tag{4.8}$$

By assumption, $T_n \to T^* < \infty$ as $n \to \infty$; hence $\int h_x^2(x,T_n)\, dx$ remains bounded. Assumption (4.7) then implies that $\int G_0(h(x,T_n))\, dx \to \infty$ as $n \to \infty$.

To continue, return to the approximate solutions h_ε. By (A.8),

$$\begin{aligned}\int_\Omega G_\varepsilon(h_\varepsilon(x,T_{n,\varepsilon}))\, dx \le{}& \int_\Omega G_\varepsilon(h_\varepsilon(x,T_{n-1,\varepsilon}))\, dx \\ &+ c_5 \int_{T_{n-1,\varepsilon}}^{T_{n,\varepsilon}} \max\left\{1, \int_\Omega h_{\varepsilon,x}^2(x,T)\, dx\right\}\, dT.\end{aligned} \tag{4.9}$$

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Using (3.25), one proves the analogue of (4.2) for all $T \in [0, T_{loc,\varepsilon}]$ and then the analogue of (4.8) for all $T \in [0, T_{n,\varepsilon}]$. Using this bound,

$$\begin{aligned} \int_{T_{n-1,\varepsilon}}^{T_{n,\varepsilon}} \int_\Omega h_{\varepsilon,x}^2(x,T)\,dxdT &\le \tfrac{4}{a_0} \int_{T_{n-1,\varepsilon}}^{T_{n,\varepsilon}} \mathcal{E}_\varepsilon(0) + K\,T + K_3\,dT \\ &= \tfrac{4}{a_0} \left[\mathcal{E}_\varepsilon(0) + K_3 + \tfrac{K}{2}\,(T_{n-1,\varepsilon} + T_{n,\varepsilon}) \right] T_{n-1,loc,\varepsilon}. \end{aligned} \tag{4.10}$$

Replacing K_3 by a larger value if necessary and using (4.10) in (4.9),

$$\begin{aligned} &\int_\Omega G_\varepsilon(h_\varepsilon(x, T_{n,\varepsilon}))\,dx \\ &\qquad \le \int_\Omega G_\varepsilon(h_\varepsilon(x, T_{n-1,\varepsilon}))\,dx + (\alpha + \beta(T_{n-1,\varepsilon} + T_{n,\varepsilon}))\,T_{n-1,loc,\varepsilon} \end{aligned} \tag{4.11}$$

for some α and β which are fixed values that depend on $|\Omega|$, the coefficients of the PDE, and (possibly) on the initial data $h_{0,\varepsilon}$. Taking $\varepsilon_k \to 0$ in the sequence $\{\varepsilon_k\}$ that was used to construct h yields

$$\int_\Omega G_0(h(x,T_n))dx \le \int_\Omega G_0(h(x,T_{n-1}))dx + (\alpha + \beta(T_{n-1} + T_n))\,T_{n-1,loc}. \tag{4.12}$$

Applying (4.12) iteratively and using that $T_k < T^*$,

$$\int_\Omega G_0(h(x,T_n))\,dx \le \int_\Omega G_0(h_0(x))\,dx + (\alpha + \beta\,2\,T^*)\,T_n. \tag{4.13}$$

Hence $\int G_0(h(x,T_n))dx < \infty$ as $n \to \infty$, finishing the proof. ∎

Under certain conditions, a bound closely related to (4.2) implies that if the solution of Theorem 1 is initially constant, then it will remain constant for all time.

THEOREM 4. *Assume the coefficients a_1 and a_2 in* (1.8) *satisfy $a_1 \ge 0$, $a_2 = 0$, and $|\Omega|^2 < a_0/|a_1|$. If the initial data are constant, $h_0 \equiv C > 0$, then the solution of Theorem* 1 *satisfies $h(x,t) = C$ for all $x \in \bar\Omega$ and all $t > 0$.*

The hypotheses of Theorem 4 correspond to the following: the equation is long-wave unstable ($a_1 > 0$), there is no nonlinear advection ($a_2 = 0$), and the domain is not "too large."

Proof. Consider the approximate solution h_ε. The definition of $\mathcal{E}_\varepsilon(T)$ combined with the linear-in-time bound (3.25) implies

$$\tfrac{a_0}{2} \int_\Omega h_{\varepsilon,x}^2(x,T)\,dx \le \mathcal{E}_\varepsilon(0) + K\,T + \tfrac{|a_1|}{2} \int_\Omega h_\varepsilon^2\,dx + |a_2|\|w\|_\infty M_\varepsilon, \tag{4.14}$$

where $M_\varepsilon = \int h_{0,\varepsilon}\,dx$. Applying Poincaré's inequality (A.2) to $v_\varepsilon = h_\varepsilon - M_\varepsilon/|\Omega|$ and using $\int h_\varepsilon^2\,dx = \int v_\varepsilon^2\,dx + M_\varepsilon^2/|\Omega|$ yields

$$\left(\tfrac{a_0}{2} - \tfrac{|a_1|\,|\Omega|^2}{2} \right) \int_\Omega h_{\varepsilon,x}^2(x,t)\,dx \le \mathcal{E}_\varepsilon(0) + K\,T_{\varepsilon,loc} + \tfrac{|a_1| M_\varepsilon^2}{2|\Omega|} + |a_2|\|w\|_\infty M_\varepsilon.$$

If $h_{0,\varepsilon} \equiv C_\varepsilon = C + \varepsilon^\theta$ and $a_2 = 0$ (hence $K = 0$), this becomes

$$\left(\tfrac{a_0}{2} - \tfrac{|a_1||\Omega|^2}{2} \right) \int_\Omega h_{\varepsilon,x}^2(x,T)\,dx \le (a_1 - |a_1|)\tfrac{C^2|\Omega|}{2}.$$

If $a_1 \ge 0$ and $|\Omega|^2 < a_0/a_1$, then $\int h_{\varepsilon,x}^2(x,T)\,dx = 0$ for all $T \in [0, T_{\varepsilon,loc}]$, and this, combined with the continuity in space and time of h_ε, implies that $h_\varepsilon \equiv C_\varepsilon$ on $Q_{T_{\varepsilon,loc}}$. Taking the sequence $\{\varepsilon_k\}$ that yields convergence to the solution h of Theorem 1, $h \equiv C$ on $Q_{T_{loc}}$. ∎

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5. Strong positivity of solutions.

PROPOSITION 5.1. *Assume the initial data h_0 satisfies $h_0(x) > 0$ for all $x \in \omega \subseteq \Omega$, where ω is an open interval. Then the weak solution h from Theorem 1 satisfies*

(1) $h(x,T) > 0$ for almost every $x \in \omega$, for all $T \in [0, T_{loc}]$;

(2) $h(x,T) > 0$ for all $x \in \omega$, for almost every $T \in [0, T_{loc}]$.

The proof of Proposition 5.1 depends on a local version of the a priori bound (3.24) of Lemma 3.1.

LEMMA 5.1. *Let $\omega \subseteq \Omega$ be an open interval and $\zeta \in C^2(\bar{\Omega})$ such that $\zeta > 0$ on ω, $\operatorname{supp} \zeta = \overline{\omega}$, and $(\zeta^4)' = 0$ on $\partial\Omega$. If $\omega = \Omega$, choose ζ such that $\zeta(-a) = \zeta(a) > 0$. Let $\xi := \zeta^4$.*

If the initial data h_0 and the time T_{loc} are as in Theorem 1, then for all $T \in [0, T_{loc}]$ the weak solution h from Theorem 1 satisfies

$$\int_\Omega \xi(x) \, \tfrac{1}{h(x,T)} \, dx < \infty. \tag{5.1}$$

The proof of Lemma 5.1 is given in Appendix A. The proof of Proposition 5.1 is essentially a combination of the proofs of Corollary 4.5 and Theorem 6.1 in [8] and is provided here for the reader's convenience.

Proof of Proposition 5.1. Choose the localizing function $\zeta(x)$ to satisfy the hypotheses of Lemma 5.1. Hence, (5.1) holds for every $T \in [0, T_{loc}]$.

First, we prove $h(x,T) > 0$ for almost every $x \in \omega$, for all $T \in [0, T_{loc}]$. Assume not. Then there is a time $T \in [0, T_{loc}]$ such that the set $\{x \mid h(x,T) = 0\} \cap \omega$ has positive measure. Then

$$\infty > \int_\Omega \xi(x) \tfrac{1}{h(x,T)} \, dx \geq \int_{\{h(\cdot,T)=0\}\cap\omega} \xi(x) \tfrac{1}{h(x,T)} \, dx = \infty.$$

This contradiction implies there can be no time at which h vanishes on a set of positive measure in ω, as desired.

Now, we prove $h(x,T) > 0$ for all $x \in \omega$, for almost every $T \in [0, T_{loc}]$. By (3.12), $h_{xx}(\cdot, T) \in L^2(\Omega)$ for almost all $T \in [0, T_{loc}]$; hence $h(\cdot, T) \in C^{3/2}(\Omega)$ for almost all $T \in [0, T_{loc}]$. Assume T_0 is such that $h(\cdot, T_0) \in C^{3/2}(\Omega)$ and $h(x_0, T_0) = 0$ at some $x_0 \in \omega$. Then there is an L such that

$$h(x,T_0) = |h(x,T_0) - h(x_0,T_0)| \leq L|x - x_0|^{3/2}.$$

Hence

$$\infty > \int_\Omega \xi(x) \tfrac{1}{h(x,T_0)} \, dx \geq \tfrac{1}{L} \int_\Omega \xi(x) |x - x_0|^{-3/2} \, dx = \infty.$$

This contradiction implies there can be no point x_0 such that $h(x_0, T_0) = 0$, as desired. Note that we used $\xi > 0$ on ω and $x_0 \in \omega$ to conclude that the integral diverges. ∎

We close our discussion with illustrations of positivity and long-time existence via numerical simulations of the initial value problem for different regimes of the PDE.

Figure 2 considers the PDE with no advection, $h_t + (h^3(h_{xxx} + 16\, h_x))_x = 0$. The PDE is translation invariant in x, and constant steady states are linearly unstable. As a result, any nonconstant behavior observed in a solution starting from constant initial data would be due to growth of round-off error. For this reason, nonconstant initial data is chosen: $h_0(x) = 0.3 + 0.02\, \cos(x) + 0.02\, \cos(2x)$. The L^2 and H^1 norms of the resulting solution appear to be converging to limiting values as time passes,

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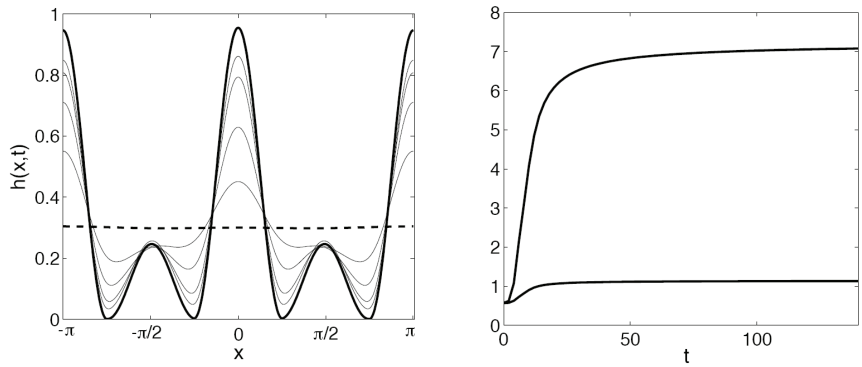

FIG. 2. *The evolution equation with no linear or nonlinear advection,* $h_t+(h^3(h_{xxx}+16\,h_x))_x = 0$*, corresponding to* $a_0 = 1$*,* $a_1 = 16$*, and* $a_2 = a_3 = 0$*. The initial data is* $h_0(x) = 0.3+0.02\,\cos(x)+ 0.02\,\cos(2x)$*. Left plot: the solution at times* $t = 0$ *(dashed line),* $t = 12, 12.5, 13, 15$ *(solid lines), and* $t = 140$ *(heavy line). Right plot: the* L^2 *and* H^1 *norms plotted as a function of time.*

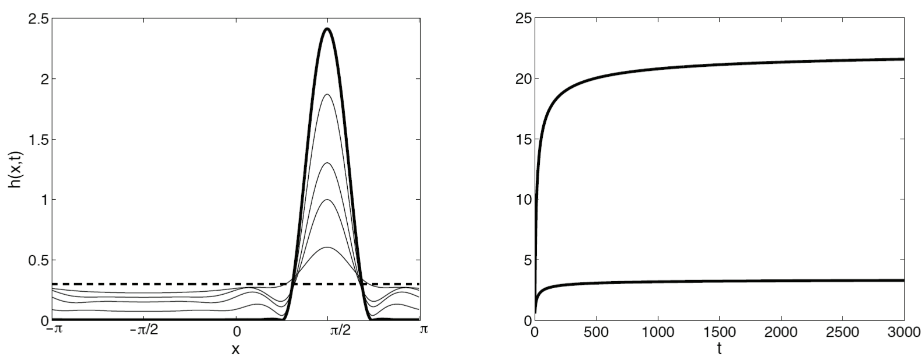

FIG. 3. *The evolution equation with nonlinear advection but no linear advection,* $h_t+(h^3(h_{xxx}+ 16\,h_x - 8\cos(x)))_x = 0$*, corresponding to* $a_0 = 1$*,* $a_1 = 16$*,* $a_2 = 8$*, and* $a_3 = 0$*. The initial data is* $h_0(x) = 0.3$*. Left plot: the solution at times* $t = 0$ *(dashed line),* $t = 0.5, 1, 2, 10$ *(solid lines), and* $t = 3000$ *(heavy line). Right plot: the* L^2 *and* H^1 *norms plotted as a function of time.*

and the long-time limit of the solution appears to be four steady state droplets of the form $a\cos(4x + \phi) + b$ for appropriate values of a, ϕ, and b. Like the PDE, the simulation shown respects the symmetry about $x = 0$ of the initial data.

Figure 3 shows the evolution from constant initial data for the PDE with nonlinear advection but no linear advection: $h_t + (h^3(h_{xxx} + 16\,h_x - 8\cos(x)))_x = 0$. The long-time limit appears to be a steady state which is zero (or nearly zero on $[-\pi, 0]$) with a droplet supported within $(0, \pi)$ and centered roughly about the midpoint ($x = \pi/2$).

Finally, Figure 4 shows the evolution resulting from the same constant initial data for the PDE with both linear and nonlinear advection: $h_t + (h^3(h_{xxx} + 16\,h_x - 8\cos(x)))_x + 3h_x = 0$. The long-time limit appears to be a strictly positive steady state.

We close by noting that the PDE considered in Figure 4 corresponds to coefficient $a_3 = 3$ in the PDE (1.8). As we increase the value of a_3 we find there appears to be a critical value past which the solution appears to converge to a time-periodic behavior rather than a steady state.

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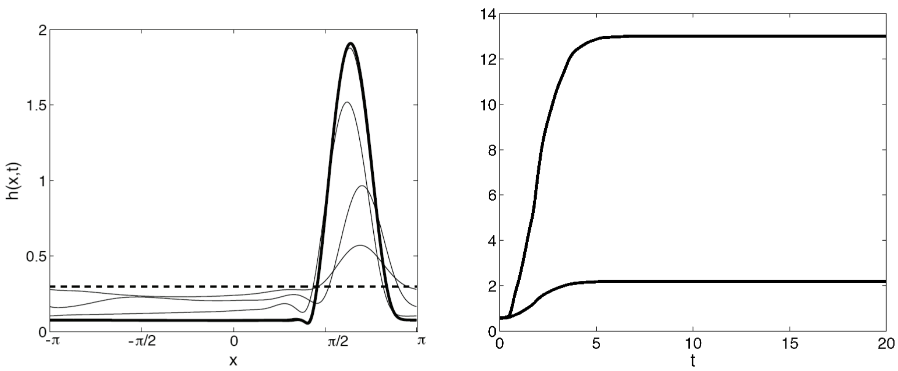

FIG. 4. *The evolution equation with both linear and nonlinear advection,* $h_t+(h^3(h_{xxx}+16\,h_x-8\cos(x)))_x+3h_x=0$*, corresponding to* $a_0=1$*,* $a_1=16$*,* $a_2=8$*, and* $a_3=3$*. The initial data is* $h_0(x)=0.3$*. Left plot: the solution at times* $t=0$ *(dashed line),* $t=0.5,1,2,4$ *(solid lines), and* $t=20$ *(heavy line). Right plot: the* L^2 *and* H^1 *norms plotted as a function of time.*

Appendix A. Proofs of a priori estimates. The first observation is that the periodic boundary conditions imply that classical solutions of (3.17) conserve mass:

$$\int_\Omega h_{\delta\varepsilon}(x,t)\,dx=\int_\Omega h_{0,\varepsilon}(x)\,dx=M_\varepsilon<\infty\quad\forall\ t>0. \tag{A.1}$$

Further, (3.21) implies $M_\varepsilon\to M=\int h_0$ as $\varepsilon,\delta\to 0$. The initial data in this article have $M>0$; hence $M_\varepsilon>0$ for δ and ε sufficiently small.

Also, we will relate the L^p norm of h to the L^p norm of its zero-mean part as follows:

$$|h(x)|\le\left|h(x)-\tfrac{M}{|\Omega|}\right|+\tfrac{M}{|\Omega|}\Longrightarrow\|h\|_p^p\le 2^{p-1}\,\|v\|_p^p+\left(\tfrac{2}{|\Omega|}\right)^{p-1}\,M^p,$$

where $v:=h-M/|\Omega|$, and we have assumed that $M\ge 0$. We will use the Poincaré inequality which holds for any zero-mean function in $H^1(\Omega)$,

$$\|v\|_p^p\le b_1\|v_x\|_p^p,\qquad 1\le p<\infty, \tag{A.2}$$

with $b_1=|\Omega|^p$.

Also used will be an interpolation inequality [27, Theorem 2.2, p. 62] for functions of zero mean in $H^1(\Omega)$:

$$\|v\|_p^p\le b_2\,\|v_x\|_2^{ap}\,\|v\|_r^{(1-a)p}, \tag{A.3}$$

where $r\ge 1$, $p\ge r$,

$$a=\tfrac{1/r-1/p}{1/r+1/2},\qquad b_2=(1+r/2)^{ap}\,.$$

It follows that for any zero-mean function v in $H^1(\Omega)$

$$\|v\|_p^p\le b_3\|v_x\|_2^p,\quad\Longrightarrow\quad\|h\|_p^p\le b_4\|h_x\|_2^p+b_5M_\varepsilon^p, \tag{A.4}$$

where

$$b_3=\begin{cases}b_1\,|\Omega|^{(2-p)/p} & \text{if}\quad 1\le p\le 2,\\ b_1^{(p+2)/2}\,b_2 & \text{if}\quad 2<p<\infty,\end{cases}\qquad b_4=2^{p-1}\,b_3,\quad b_5=\left(\tfrac{2}{|\Omega|}\right)^{p-1}.$$

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To see that (A.4) holds, consider two cases. If $1 \le p < 2$, then by (A.2), $\|v\|_p$ is controlled by $\|v_x\|_p$. By the Hölder inequality, $\|v_x\|_p$ is then controlled by $\|v_x\|_2$. If $p > 2$, then by (A.3), $\|v\|_p$ is controlled by $\|v_x\|_2^a \|v\|_2^{1-a}$, where $a = 1/2 - 1/p$. By the Poincaré inequality, $\|v\|_2^{1-a}$ is controlled by $\|v_x\|_2^{1-a}$.

Proof of Lemma 3.1. In the following, we denote the classical solution $h_{\delta\varepsilon}$ by h whenever there is no chance of confusion.

To prove the bound (3.23) one starts by multiplying (3.17) by $-h_{xx}$, integrating over Q_T, and using the periodic boundary conditions (3.18), which yields

$$\begin{aligned} \text{(A.5)} \qquad & \tfrac{1}{2}\int_\Omega h_x^2(x,T)\,dx + a_0 \iint\limits_{Q_T} f_{\delta\varepsilon}(h) h_{xxx}^2\,dxdt \\ & = \tfrac{1}{2}\int_\Omega h_{0,\varepsilon,x}^2(x)\,dx - a_1 \iint\limits_{Q_T} f_\varepsilon(h) h_x h_{xxx}\,dxdt + \delta a_1 \iint\limits_{Q_T} h_{xx}^2\,dxdt \\ & \quad - a_2 \iint\limits_{Q_T} f_{\delta\varepsilon}(h) w' h_{xxx}\,dxdt. \end{aligned}$$

By Cauchy and Young inequalities, due to (A.2)–(A.4), it follows from (A.5) that

$$\begin{aligned} \text{(A.6)} \qquad & \tfrac{1}{2}\int_\Omega h_x^2(x,T)\,dx + \tfrac{a_0}{2} \iint\limits_{Q_T} f_{\delta\varepsilon}(h) h_{xxx}^2\,dxdt \\ & \le \tfrac{1}{2}\int_\Omega h_{0,\varepsilon,x}^2\,dx + c_3 \iint\limits_{Q_T} h_{xx}^2\,dxdt + c_4 \int_0^T \max\left\{1, \left(\int_\Omega h_x^2\,dx\right)^3\right\}\,dt, \end{aligned}$$

where

$$c_1 = b_2^2/8 + b_4/2, c_2 = M_{\delta\varepsilon}^6\ b_5/2, c_3 = \tfrac{a_1^2}{2a_0} + \delta|a_1|,$$

$$c_4 = \tfrac{a_1^2}{a_0} c_1 + \tfrac{a_2^2}{a_0}\|w'\|_\infty^2 b_4 + \tfrac{a_1^2}{a_0} c_2 + \tfrac{a_2^2}{a_0}\|w'\|_\infty^2 b_5 M_{\delta\varepsilon}^3 + \delta \tfrac{a_2^2}{a_0}\|w'\|_2^2.$$

Now, multiplying (3.17) by $G'_{\delta\varepsilon}(h)$, integrating over Q_T, and using the periodic boundary conditions (3.18), we obtain

$$\begin{aligned} \text{(A.7)} \qquad & \int_\Omega G_{\delta\varepsilon}(h(x,T))\,dx + a_0 \iint\limits_{Q_T} h_{xx}^2\,dxdt = \int_\Omega G_{\delta\varepsilon}(h_{0,\varepsilon})dx + a_1 \iint\limits_{Q_T} h_x^2 dxdt \\ & \qquad - a_3 \iint\limits_{Q_T} (G_{\delta\varepsilon}(h))_x\,dxdt + a_2 \iint\limits_{Q_T} w' h_x\,dxdt. \end{aligned}$$

By the periodic boundary conditions, we deduce

$$\begin{aligned} & \int_\Omega G_{\delta\varepsilon}(h(x,T))\,dx + a_0 \iint\limits_{Q_T} h_{xx}^2\,dxdt \\ \text{(A.8)} \qquad & \leqslant \int_\Omega G_{\delta\varepsilon}(h_{0,\varepsilon})\,dx + c_5 \int_0^T \max\left\{1, \int_\Omega h_x^2(x,t)\,dx\right\}\,dt, \end{aligned}$$

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where $c_5 = |a_1| + |a_2|\|w'\|_2$. Further, from (A.6) and (A.8) we find

$$\int_\Omega h_x^2\,dx + \tfrac{2c_3}{a_0}\int_\Omega G_{\delta\varepsilon}(h)\,dx + a_0 \iint\limits_{Q_T} f_{\delta\varepsilon}(h)h_{xxx}^2\,dxdt \le \int_\Omega h_{0,\varepsilon,x}^2 dx$$

$$+ \tfrac{2c_3}{a_0}\int_\Omega G_{\delta\varepsilon}(h_{0,\varepsilon})dx + c_6\int_0^T \max\left\{1, \left(\int_\Omega h_x^2(x,t)\,dx\right)^3\right\}dt, \tag{A.9}$$

where $c_6 = 2c_3c_5/a_0 + 2c_4$. Applying the nonlinear Grönwall lemma [15] to

$$v(T) \le v(0) + c_6\int_0^T \max\{1, v^3(t)\}\,dt$$

with $v(t) = \int (h_x^2(x,t) + 2c_3/a_0\, G_{\delta\varepsilon}(h(x,t)))\,dx$ yields

$$\int_\Omega h_x^2(x,t) + 2\tfrac{c_3}{a_0}G_{\delta\varepsilon}(h(x,t))\,dx \tag{A.10}$$

$$\le \sqrt{2}\max\left\{1, \int_\Omega (h_{0,\varepsilon,x}^2(x) + 2\tfrac{c_3}{a_0}G_{\delta\varepsilon}(h_{0,\varepsilon}(x)))\,dx\right\} = K_{\delta\varepsilon} < \infty$$

for all $t \in [0, T_{\delta\varepsilon,loc}]$, where

$$T_{\delta\varepsilon,loc} := \tfrac{1}{4c_6}\min\left\{1, \left(\int_\Omega (h_{0,\varepsilon,x}^2(x) + 2\tfrac{c_3}{a_0}G_{\delta\varepsilon}(h_{0,\varepsilon}(x)))\,dx\right)^{-2}\right\}. \tag{A.11}$$

Using the $\delta \to 0, \varepsilon \to 0$ convergence of the initial data and the choice of $\theta \in (0, 2/5)$ (see (3.21)) as well as the assumption that the initial data h_0 has finite entropy (3.11), the times $T_{\delta\varepsilon,loc}$ converge to a positive limit, and the upper bound K in (A.10) can be taken finite and independent of δ and ε for δ and ε sufficiently small. (We refer the reader to the end of the proof of Lemma 5.1 in this appendix for a fuller explanation of a similar case.) Therefore there exists $\delta_0 > 0$ and $\varepsilon_0 > 0$ and K such that the bound (A.10) holds for all $0 \le \delta < \delta_0$ and $0 < \varepsilon < \varepsilon_0$ with K replacing $K_{\delta\varepsilon}$ and for all

$$0 \le t \le T_{loc} := \tfrac{9}{10}\lim_{\varepsilon\to 0,\delta\to 0} T_{\delta\varepsilon,loc}. \tag{A.12}$$

Using the uniform bound on $\int h_x^2$ that (A.10) provides, one can find a uniform-in-δ-and-ε bound for the right-hand side of (A.9), yielding the desired a priori bound (3.23). Similarly, one can find a uniform-in-δ-and-ε bound for the right-hand side of (A.8), yielding the desired a priori bound (3.24).

To prove the bound (3.25), multiply (3.17) by $-a_0h_{xx} - a_1h - a_2w$, integrate over Q_T, integrate by parts, use the periodic boundary conditions (3.18), and use the mass conservation (see (A.1)) to find

$$\mathcal{E}_{\delta\varepsilon}(T) + \iint\limits_{Q_T} f_{\delta\varepsilon}(h)(a_0h_{xxx} + a_1h_x + a_2w'(x))^2\,dxdt$$

$$\le \mathcal{E}_{\delta\varepsilon}(0) + |a_2a_3|\|w'\|_\infty\left(|\Omega|^{3/2}\sqrt{K_1} + M\right)T. \tag{A.13}$$

Hence the desired bound (3.25) is obtained if the constant

$$K_3 = |a_2a_3|\|w'\|_\infty(|\Omega|^{3/2}\sqrt{K_1} + M).$$

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The time T_{loc} and the constants K_1, K_2, and K_3 are determined by δ_0, ε_0, a_0, a_1, a_2, w', $|\Omega|$, and h_0. $\square$

Proof of Lemma 3.2. In the following, we denote the positive, classical solution h_ε by h whenever there is no chance of confusion.

Multiplying (3.17) by $(G_\varepsilon^{(\alpha)}(h))'$, integrating over Q_T, taking $\delta \to 0$, and using the periodic boundary conditions (3.18) yields

$$\begin{aligned}(\text{A.14}) \qquad &\int_\Omega G_\varepsilon^{(\alpha)}(h(x,T))\,dx + a_0 \iint\limits_{Q_T} h^\alpha h_{xx}^2\,dxdt + a_0\tfrac{\alpha(1-\alpha)}{3} \iint\limits_{Q_T} h^{\alpha-2}h_x^4\,dxdt \\ &= \int_\Omega G_\varepsilon^{(\alpha)}(h_{0\varepsilon})\,dx + a_1 \iint\limits_{Q_T} h^\alpha h_x^2\,dxdt - \tfrac{a_2}{\alpha+1} \iint\limits_{Q_T} h^{\alpha+1}w''\,dxdt.\end{aligned}$$

Case 1. $0 < \alpha < 1$. The coefficient multiplying $\iint h^{\alpha-2}h_x^4$ in (A.14) is positive and can therefore be used to control the term $\iint h^\alpha h_x^2$ on the right-hand side of (A.14). Specifically, using the Cauchy–Schwarz inequality and the Cauchy inequality,

$$(\text{A.15}) \quad a_1 \iint\limits_{Q_T} h^\alpha h_x^2\,dxdt \leqslant \tfrac{a_0\alpha(1-\alpha)}{6} \iint\limits_{Q_T} h^{\alpha-2}h_x^4\,dxdt + \tfrac{3a_1^2}{2a_0\alpha(1-\alpha)} \iint\limits_{Q_T} h^{\alpha+2}\,dxdt.$$

Using the bound (A.15) in (A.14) yields

$$\begin{aligned}(\text{A.16}) \qquad &\int_\Omega G_\varepsilon^{(\alpha)}(h(x,T))\,dx + a_0 \iint\limits_{Q_T} h^\alpha h_{xx}^2\,dxdt + a_0\tfrac{\alpha(1-\alpha)}{6} \iint\limits_{Q_T} h^{\alpha-2}h_x^4\,dxdt \\ &\leq \int_\Omega G_\varepsilon^{(\alpha)}(h_{0\varepsilon})\,dx + \tfrac{3a_1^2}{2a_0\alpha(1-\alpha)} \iint\limits_{Q_T} h^{\alpha+2}\,dxdt + \tfrac{|a_2|\|w''\|_\infty}{\alpha+1} \iint\limits_{Q_T} h^{\alpha+1}dxdt.\end{aligned}$$

By (A.4),

$$\begin{aligned}&\int_\Omega G_\varepsilon^{(\alpha)}(h(x,T))\,dx + a_0 \iint\limits_{Q_T} h^\alpha h_{xx}^2\,dxdt + a_0\tfrac{\alpha(1-\alpha)}{6} \iint\limits_{Q_T} h^{\alpha-2}h_x^4\,dxdt \\ (\text{A.17}) \qquad &\leq \int_\Omega G_\varepsilon^{(\alpha)}(h_{0\varepsilon})\,dx + d_1 \int_0^T \max\left\{1, \left(\int_\Omega h_x^2\,dx\right)^{\frac{\alpha}{2}+1}\right\} dt,\end{aligned}$$

where

$$d_1 = b_4 \left(\tfrac{3a_1^2}{2a_0\alpha(1-\alpha)} + \tfrac{|a_2|\|w''\|_\infty}{1+\alpha}\right) + b_5 \left(\tfrac{3a_1^2}{2a_0\alpha(1-\alpha)}\, M_\varepsilon^{\alpha+2} + \tfrac{|a_2|\|w''\|_\infty}{1+\alpha}\, M_\varepsilon^{\alpha+1}\right).$$

Using the Cauchy inequality in (A.9) and taking $\delta \to 0$ yields

$$\begin{aligned}(\text{A.18}) \qquad &\int_\Omega h_x^2\,dx + a_0 \iint\limits_{Q_T} f_\varepsilon(h) h_{xxx}^2\,dxdt \\ &\leq \int_\Omega h_{0\varepsilon,x}^2\,dx + \tfrac{2a_1^2}{a_0} \iint\limits_{Q_T} h^3 h_x^2\,dxdt + \tfrac{2a_2^2\|w'\|_\infty^2}{a_0} \iint\limits_{Q_T} h^3\,dxdt.\end{aligned}$$

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Applying the Cauchy–Schwarz inequality and (A.4) yields

$$\int_\Omega h_x^2\,dx + a_0 \iint\limits_{Q_T} f_\varepsilon(h) h_{xxx}^2\,dxdt \le \int_\Omega h_{0\varepsilon,x}^2\,dx$$
$$+ \tfrac{a_0\alpha(1-\alpha)}{6} \iint\limits_{Q_T} h^{\alpha-2} h_x^4\,dxdt + d_2 \int_0^T \max\left\{1, \left(\int_\Omega h_x^2\,dx\right)^{4-\frac{\alpha}{2}}\right\}\,dt,$$

where

$$d_2 = b_4 \left(\tfrac{6a_1^4}{a_0^3\alpha(1-\alpha)} + \tfrac{2a_2^2}{a_0}\|w'\|_\infty^2\right) + b_5\left(\tfrac{6a_1^4}{a_0^3\alpha(1-\alpha)}\, M_\varepsilon^{8-\alpha} + \tfrac{2a_2^2}{a_0}\|w'\|_\infty^2\, M_\varepsilon^3\right).$$

Adding $\int G_\varepsilon^{(\alpha)}(h(x,T))$ to both sides of (A.18), $a_0 \iint h^\alpha h_{xx}^2$ to the resulting right-hand side, and using (A.17), we obtain

$$\text{(A.19)}\qquad \int_\Omega h_x^2(x,T)\,dx + \int_\Omega G_\varepsilon^{(\alpha)}(h(x,T))\,dx + a_0 \iint\limits_{Q_T} f_\varepsilon(h) h_{xxx}^2\,dxdt$$
$$\le \int_\Omega h_{0\varepsilon,x}^2\,dx + \int_\Omega G_\varepsilon^{(\alpha)}(h_{0\varepsilon})\,dx + d_3 \int_0^T \max\left\{1, \left(\int_\Omega h_x^2\,dx\right)^{4-\frac{\alpha}{2}}\right\},$$

where $d_3 = d_1 + d_2$. Applying the nonlinear Grönwall lemma [15] to

$$v(T) \le v(0) + d_3 \int_0^T \max\{1, v^{4-\alpha/2}(t)\}\,dt$$

with $v(T) = \int (h_x^2(x,T) + G_\varepsilon^{(\alpha)}(h(x,T)))\,dx$ yields

$$\text{(A.20)}\qquad \int_\Omega (h_x^2(x,T) + G_\varepsilon^{(\alpha)}(h(x,T)))\,dx$$
$$\le 4^{\frac{1}{6-\alpha}} \max\left\{1, \int_\Omega ({h_{0,\varepsilon}}_x^2(x) + G_\varepsilon^{(\alpha)}(h_{0,\varepsilon}(x)))\,dx\right\} = K_\varepsilon < \infty$$

for all T:

$$0 \le T \le T_{\varepsilon,loc}^{(\alpha)} := \tfrac{1}{d_3(6-\alpha)} \min\left\{1, \left(\int_\Omega ({h_{0,\varepsilon}}_x^2(x) + G_\varepsilon^{(\alpha)}(h_{0,\varepsilon}(x)))\,dx\right)^{-\frac{6-\alpha}{2}}\right\}.$$

The bound (A.20) holds for all $0 < \varepsilon < \varepsilon_0$, where ε_0 is from Lemma 3.1, and for all $t \le \min\{T_{loc}, T_{\varepsilon,loc}^{(\alpha)}\}$, where T_{loc} is from Lemma 3.1.

Using the $\varepsilon \to 0$ convergence of the initial data and the choice of $\theta \in (0, 2/5)$ (see (3.21)) as well as the assumption that the initial data h_0 has finite α-entropy (3.27), the times $T_{\varepsilon,loc}^{(\alpha)}$ converge to a positive limit and the upper bound K_ε in (A.20) can be taken finite and independent of ε. (We refer the reader to the end of the proof of Lemma 5.1 in this appendix for a fuller explanation of a similar case.) Therefore there exists $\varepsilon_0^{(\alpha)}$ and K such that the bound (A.20) holds for all $0 < \varepsilon < \varepsilon_0^{(\alpha)}$ with K

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replacing K_ε and for all

$$0 \le t \le T_{loc}^{(\alpha)} := \min\left\{T_{loc}, \tfrac{9}{10}\lim_{\varepsilon\to 0} T_{\varepsilon,loc}^{(\alpha)}\right\}, \tag{A.21}$$

where T_{loc} is the time from Lemma 3.1. Also, without loss of generality, $\varepsilon_0^{(\alpha)}$ can be taken to be less than or equal to the ε_0 from Lemma 3.1.

Using the uniform bound on $\int h_x^2$ that (A.20) provides, one can find a uniform-in-ε bound for the right-hand side of (A.17), yielding the desired bound

$$\int_\Omega G_\varepsilon^{(\alpha)}(h(x,T))\,dx + a_0 \iint\limits_{Q_T} h^\alpha h_{xx}^2\,dxdt + a_0\tfrac{\alpha(1-\alpha)}{6}\iint\limits_{Q_T} h^{\alpha-2}h_x^4\,dxdt \le K_1, \tag{A.22}$$

which holds for all $0<\varepsilon<\varepsilon_0^{(\alpha)}$ and all $0 \le T \le T_{loc}^{(\alpha)}$.

It remains to argue that (A.22) implies that for all $0<\varepsilon<\varepsilon_0^{(\alpha)}$ that $h_\varepsilon^{\alpha/2+1}$ and $h_\varepsilon^{\alpha/4+1/2}$ are contained in balls in $L^2(0,T;H^2(\Omega))$ and $L^2(0,T;W_4^1(\Omega))$, respectively. It suffices to show that

$$\iint\limits_{Q_T}\left(h_\varepsilon^{\alpha/2+1}\right)_{xx}^2\,dxdt \le K, \qquad \iint\limits_{Q_T}\left(h_\varepsilon^{\alpha/4+1/2}\right)_x^4\,dxdt \le K$$

for some K that is independent of ε and T. The integral $\iint (h_\varepsilon^{\alpha/2+1})_{xx}^2$ is a linear combination of $\iint h^{\alpha-2}h_x^4$, $\iint h^{\alpha-1}h_x^2 h_{xx}$, and $\iint h^\alpha h_{xx}^2$. Integration by parts and the periodic boundary conditions imply

$$\tfrac{1-\alpha}{3}\iint\limits_{Q_T} h^{\alpha-2}h_x^4\,dxdt = \iint\limits_{Q_T} h^{\alpha-1}h_x^2 h_{xx}\,dxdt. \tag{A.23}$$

Hence $\iint (h_\varepsilon^{\alpha/2+1})_{xx}^2$ is a linear combination of $\iint h^{\alpha-2}h_x^4$, and $\iint h^\alpha h_{xx}^2$. By (A.22), the two integrals are uniformly bounded independent of ε and T; hence $\iint (h_\varepsilon^{\alpha/2+1})_{xx}^2$ is as well, yielding the first part of (3.29).

The uniform bound of $\iint (h_\varepsilon^{\alpha/4+1/2})_x^4$ follows immediately from the uniform bound of $\iint h^{\alpha-2}h_x^4$, yielding the second part of (3.29).

Case 2. $-\frac{1}{2}<\alpha<0$. For $\alpha<0$ the coefficient multiplying $\iint h^{\alpha-2}h_x^4$ in (A.14) is negative. However, we will show that if $\alpha > -1/2$, then one can replace this coefficient with a positive coefficient while also controlling the term $\iint h^\alpha h_x^2$ on the right-hand side of (A.14).

Applying the Cauchy–Schwarz inequality to the right-hand side of (A.23), dividing by $\sqrt{\iint h^{\alpha-2}h_x^4}$, and squaring both sides of the resulting inequality yields

$$\iint\limits_{Q_T} h^{\alpha-2}h_x^4\,dxdt \le \tfrac{9}{(1-\alpha)^2}\iint\limits_{Q_T} h^\alpha h_{xx}^2\,dxdt \qquad \forall \alpha<1. \tag{A.24}$$

Using (A.24) in (A.14) yields

$$\begin{aligned}\int_\Omega G_\varepsilon^{(\alpha)}(h(x,T))\,dx &+ a_0\tfrac{1+2\alpha}{1-\alpha}\iint\limits_{Q_T} h^\alpha h_{xx}^2\,dxdt \\ &\le \int_\Omega G_\varepsilon^{(\alpha)}(h_{0\varepsilon})\,dx + a_1\iint\limits_{Q_T} h^\alpha h_x^2\,dxdt + \tfrac{|a_2|}{\alpha+1}\|w''\|_\infty \iint\limits_{Q_T} h^{\alpha+1}\,dxdt.\end{aligned} \tag{A.25}$$

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Note that if $\alpha > -1/2$, then all the terms on the left-hand side of (A.25) are positive. We now control the term $\iint h^\alpha h_x^2$ on the right-hand side of (A.25).

By integration by parts and the periodic boundary conditions,

$$\iint\limits_{Q_T} h^\alpha h_x^2 \, dxdt = -\tfrac{1}{1+\alpha} \iint\limits_{Q_T} h^{\alpha+1} h_{xx} \, dxdt. \tag{A.26}$$

Applying the Cauchy inequality to (A.26) yields

$$a_1 \iint\limits_{Q_T} h^\alpha h_x^2 \, dxdt \leq \iint\limits_{Q_T} \left(\tfrac{a_0(1+2\alpha)}{2(1-\alpha)} h^\alpha h_{xx}^2 + \tfrac{a_1^2(1-\alpha)}{2a_0(1+2\alpha)(1+\alpha)^2} h^{\alpha+2} \right) dxdt. \tag{A.27}$$

Using inequality (A.27) in (A.25) yields

$$\begin{aligned} &\int_\Omega G_\varepsilon^{(\alpha)}(h(x,T)) \, dx + a_0 \tfrac{1+2\alpha}{2(1-\alpha)} \iint\limits_{Q_T} h^\alpha h_{xx}^2 \, dxdt \\ &\leq \int_\Omega G_\varepsilon^{(\alpha)}(h_{0\varepsilon}) \, dx + \iint\limits_{Q_T} \left(\tfrac{a_1^2(1-\alpha)}{2a_0(1+2\alpha)(1+\alpha)^2} h^{\alpha+2} + \tfrac{|a_2|}{\alpha+1} \|w''\|_\infty h^{\alpha+1} \right) dxdt. \end{aligned} \tag{A.28}$$

Adding

$$\tfrac{a_0(1+2\alpha)(1-\alpha)}{36} \iint\limits_{Q_T} h^{\alpha-2} h_x^4 \, dxdt$$

to both sides of (A.28) and using the inequality (A.24) yields

$$\begin{aligned} &\int_\Omega G_\varepsilon^{(\alpha)}(h(x,T)) \, dx + a_0 \tfrac{(1+2\alpha)}{4(1-\alpha)} \iint\limits_{Q_T} h^\alpha h_{xx}^2 \, dxdt \\ &\quad + \tfrac{a_0(1+2\alpha)(1-\alpha)}{36} \iint\limits_{Q_T} h^{\alpha-2} h_x^4 \, dxdt \leq \int_\Omega G_\varepsilon^{(\alpha)}(h_{0\varepsilon}) \, dx \\ &\quad + \tfrac{a_1^2(1-\alpha)}{2a_0(1+2\alpha)(1+\alpha)^2} \iint\limits_{Q_T} h^{\alpha+2} \, dxdt + \tfrac{|a_2|}{\alpha+1} \|w''\|_\infty \iint\limits_{Q_T} h^{\alpha+1} \, dxdt. \end{aligned} \tag{A.29}$$

Using (A.29) and (A.4) yields

$$\begin{aligned} &\int_\Omega G_\varepsilon^{(\alpha)}(h(x,T)) \, dx + \iint\limits_{Q_T} \left(\tfrac{a_0(1+2\alpha)}{4(1-\alpha)} h^\alpha h_{xx}^2 + \tfrac{a_0(1+2\alpha)(1-\alpha)}{36} h^{\alpha-2} h_x^4 \right) dxdt \\ &\leq \int_\Omega G_\varepsilon^{(\alpha)}(h_{0\varepsilon}) \, dx + e_1 \int_0^T \max\left\{ 1, \left(\int_\Omega h_x^2 \, dx \right)^{\frac{\alpha}{2}+1} \right\} dt, \end{aligned} \tag{A.30}$$

where

$$e_1 = b_4 \left(\tfrac{a_1^2(1-\alpha)}{2a_0(1+2\alpha)(1+\alpha)^2} + \tfrac{|a_2|}{\alpha+1} \|w''\|_\infty \right) + b_5 \left(\tfrac{a_1^2(1-\alpha)}{2a_0(1+2\alpha)(1+\alpha)^2} \, M_\varepsilon^{\alpha+2} + \tfrac{|a_2|}{\alpha+1} \|w''\|_\infty \, M_\varepsilon^{\alpha+1} \right).$$

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Recall the bound (A.18). As before, by the Cauchy inequality,

$$\frac{2a_1^2}{a_0}\iint\limits_{Q_T} h^3 h_x^2\,dxdt \le \frac{a_0(1+2\alpha)(1-\alpha)}{36}\iint\limits_{Q_T} h^{\alpha-2}h_x^4\,dxdt + \frac{36a_1^4}{a_0^3(1+2\alpha)(1-\alpha)}\iint\limits_{Q_T} h^{8-\alpha}\,dxdt. \tag{A.31}$$

Using (A.31) in (A.18) yields

$$\int_\Omega h_x^2\,dx + a_0\iint\limits_{Q_T} f_\varepsilon(h)h_{xxx}^2\,dxdt \le \int_\Omega h_{0\varepsilon,x}^2\,dx + \frac{a_0(1+2\alpha)(1-\alpha)}{36}\iint\limits_{Q_T} h^{\alpha-2}h_x^4\,dxdt + e_2\int_0^T \max\left\{1,\left(\int_\Omega h_x^2\,dx\right)^{4-\frac{\alpha}{2}}\right\}\,dt,$$

where $e_2 = b_4\big(\frac{36a_1^4}{a_0^3(1+2\alpha)(1-\alpha)} + \frac{2a_2^2}{a_0}\|w'\|_\infty^2\big) + b_5\big(\frac{36a_1^4}{a_0^3(1+2\alpha)(1-\alpha)}\,M_\varepsilon^{8-\alpha} + \frac{2a_2^2}{a_0}\|w'\|_\infty^2\,M_\varepsilon^3\big)$. Just as (A.17) and (A.18) yielded (A.19), inequality (A.30) combined with the above inequality yields

$$\int_\Omega h_x^2(x,T)\,dx + \int_\Omega G_\varepsilon^{(\alpha)}(h(x,T))\,dx + a_0\iint\limits_{Q_T} f_\varepsilon(h)h_{xxx}^2\,dxdt \le \int_\Omega h_{0\varepsilon,x}^2\,dx + \int_\Omega G_\varepsilon^{(\alpha)}(h_{0\varepsilon})\,dx + e_3\int_0^T \max\left\{1,\left(\int_\Omega h_x^2\,dx\right)^{4-\frac{\alpha}{2}}\right\}, \tag{A.32}$$

where $e_3 = e_1 + e_2$. The rest of the proof now continues as in the $0<\alpha<1$ case. Specifically, one finds a bound

$$\int_\Omega (h_x^2(x,T) + G_\varepsilon^{(\alpha)}(h(x,T)))\,dx \le 4^{\frac{1}{6-\alpha}}\max\left\{1,\int_\Omega ({h_{0,\varepsilon,x}}^2(x) + G_\varepsilon^{(\alpha)}(h_{0,\varepsilon}(x)))\,dx\right\} = K_\varepsilon < \infty \tag{A.33}$$

for all T:

$$0 \le T \le T_{\varepsilon,loc}^{(\alpha)} := \frac{1}{e_3(6-\alpha)}\min\left\{1,\left(\int_\Omega ({h_{0,\varepsilon,x}}^2(x) + G_\varepsilon^{(\alpha)}(h_{0,\varepsilon}(x)))\,dx\right)^{-\frac{6-\alpha}{2}}\right\}.$$

The time $T_{loc}^{(\alpha)}$ is defined as in (A.21), and the uniform bound (A.33) used to bound the right-hand side of (A.30) yields the desired bound

$$\int_\Omega G_\varepsilon^{(\alpha)}(h(x,T))\,dx + \frac{a_0(1+2\alpha)}{4(1-\alpha)}\iint\limits_{Q_T} h^\alpha h_{xx}^2\,dxdt + \frac{a_0(1+2\alpha)(1-\alpha)}{36}\iint\limits_{Q_T} h^{\alpha-2}h_x^4\,dxdt \le K_2. \qquad \square \tag{A.34}$$

Proof of Lemma 5.1. In the following, we denote the positive, classical solution h_ε constructed in Lemma 3.3 by h (whenever there is no chance of confusion).

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Recall the entropy function $G_{\delta\varepsilon}(z)$ defined by (3.22). Multiplying (3.17) by $\xi(x)G'_{\delta\varepsilon}(h_{\delta\varepsilon})$, taking $\delta \to 0$, and integrating over Q_T yields

$$
\begin{aligned}
\int_\Omega \xi(x)G_\varepsilon(h(x,T))dx - \int_\Omega \xi(x)G_\varepsilon(h_{0,\varepsilon})dx &= -a_3 \iint\limits_{Q_T} \xi(x)G'_\varepsilon(h)h_x dxdt \\
&+ \iint\limits_{Q_T} f_\varepsilon(h)(a_0 h_{xxx} + a_1 h_x + a_2 w')(\xi' G'_\varepsilon(h) + \xi G''_\varepsilon(h)h_x)\, dxdt \\
= a_3 \iint\limits_{Q_T} \xi' G_\varepsilon(h)\, dxdt &+ \iint\limits_{Q_T} \xi' f_\varepsilon(h) G'_\varepsilon(h)(a_0 h_{xxx} + a_1 h_x + a_2 w')\, dxdt \\
&+ \iint\limits_{Q_T} \xi h_x (a_0 h_{xxx} + a_1 h_x + a_2 w')\, dxdt =: I_1 + I_2 + I_3.
\end{aligned}
\tag{A.35}
$$

One easily finds that for all $\varepsilon > 0$ and all $z \geq 0$

$$
|f_\varepsilon(z)G'_\varepsilon(z)| \leqslant \tfrac{1}{2}z,\ \ |f'_\varepsilon(z)G'_\varepsilon(z)| \leqslant 2,
$$

$$
\left| \int_0^z f_\varepsilon(s)G'_\varepsilon(s)\, ds \right| \leq \tfrac{1}{2}z^2 + \tfrac{3}{5} \quad \text{if } 0 < \varepsilon < (\sqrt{33} - 3)/4.
$$

Using these bounds and recalling $\xi = \zeta^4$, we bound $|I_2|$:

$$
\begin{aligned}
|I_2| \leq \iint\limits_{Q_T} &\left(\tfrac{a_0}{2}\zeta^4 h_{xx}^2 + \gamma_1 \left[\zeta^2\zeta_x^2 + \zeta^3|\zeta_{xx}| + \zeta_x^4 + \zeta^2\zeta_{xx}^2 \right] \left(h^2 + h_x^2 \right) \right) dxdt \\
&+ 2|a_2|\|w'\|_\infty \iint\limits_{Q_T} \zeta^3|\zeta_x| h\ dxdt + \tfrac{3}{5}|a_1| \iint\limits_{Q_T} |\xi''|\ dxdt,
\end{aligned}
\tag{A.36}
$$

where $\gamma_1 = \max\{102a_0, 6|a_1|\}$ and $0 < \varepsilon < (\sqrt{33} - 3)/4$. Now, integrating by parts in I_3, we deduce

$$
\begin{aligned}
I_3 + a_0 \iint\limits_{Q_T} \xi h_{xx}^2\ dxdt \leq \gamma_2 &\iint\limits_{Q_T} \left[\zeta^2\zeta_x^2 + \zeta^3|\zeta_{xx}| + \zeta^4 \right] h_x^2\ dxdt \\
&+ 4|a_2| \left(\|w'\|_\infty + \|w''\|_\infty \right) \iint\limits_{Q_T} \left(\zeta^3|\zeta_x| + \zeta^4 \right) h\ dxdt,
\end{aligned}
\tag{A.37}
$$

where $\gamma_2 = \max\{6a_0, |a_1|\}$. Using bounds (A.36) and (A.37), we obtain that

$$
\int_\Omega \xi G_\varepsilon(h_\varepsilon(x,T))\, dx \leqslant \int_\Omega \xi G_\varepsilon(h_{0\varepsilon})\, dx + C,
\tag{A.38}
$$

where $C > 0$ is independent of $\varepsilon > 0$. Using the fact that θ was chosen so that $\theta < 2/5 < 1/2$, we have $|\xi(x)\, G_\varepsilon(h_{0\varepsilon}(x))| \leq \xi(x)(G_0(h_0(x)) + c) \leq C(G_0(h_0(x)) + c)$ almost everywhere in x and for all $\varepsilon < \varepsilon_0$. To finish the proof, we apply Fatou's lemma to the left-hand side and the Lebesgue dominated convergence theorem to the right-hand side of (A.38). $\square$

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Appendix B. Results used from functional analysis.

Lemma B.1 (see [28]). *Suppose that X, Y, and Z are Banach spaces, $X \Subset Y \subset Z$, and X and Z are reflexive. Then the embedding $\{u \in L^{p_0}(0,T;\, X) : \partial_t u \in L^{p_1}(0,T;Z), 1 < p_i < \infty, i = 0,1\} \Subset L^{p_0}(0,T;Y)$ is compact.*

Lemma B.2 (see [38]). *Suppose that X, Y, and Z are Banach spaces and $X \Subset Y \subset Z$. Then the embedding $\{u \in L^{\infty}(0,T;X) : \partial_t u \in L^{p}(0,T;Z),\ p > 1\} \Subset C(0,T;Y)$ is compact.*

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