# Problems

April 29, 2014

**Problem.** (Jan 10 #3) Describe all meromorphic functions $f(z)$ in the complex plane with a simple pole at $z = 1$, a simple zero at $z = -1$ and for which:

$$|f(z)| \leq M\,|z|\,,\,|z| \geq 2$$

for some $M > 0$

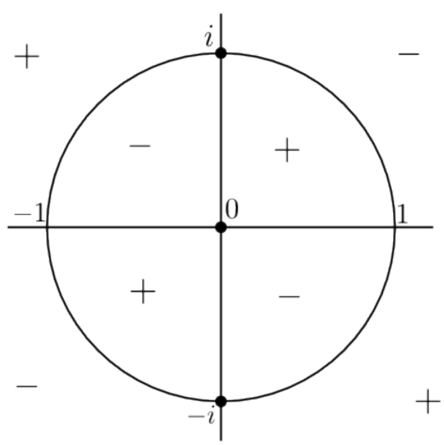
**Problem.** (Jan 11 #3) The function $f$ is analytic in the whole plane with positive imaginary part. What can it be? What if all you know is that the imaginary part of $f$ tends to 0 at $\infty$?

**Problem.** (Jan 07 #4) What is the most general entire function that takes each complex value once and only once in $\mathbb{C}$? Give a complete proof for full marks.

**Problem.** (Sept 01 #4) Suppose that $f(z)$ is entire and has n simple zeros at $z_1, z_2, \ldots, z_n$.
Part 1 Suppose $|f(z)| \leq k|z|^m + L$ for some $m$. What is $m$?
Part 2 What is the most general such function $f(z)$?
Part 3 Suppose it is known that $|f(z)| \leq Ae^{|z|^{3/2}}$, what is $f(z)$?

**Problem.** (Jan 03 #4) Let $f$ be an entire function and $n$ a positive integer. Show that there is an entire function $g$ such that $g^n = f$ if and only if the orders of zeros of $f$ are divisible by $n$.

**Problem.** (Jan 11 #5) The picture shows what the function $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ does to the plane. The values 0 at 0, 1 at $\pm 1$, and $\infty$ at $\pm i$ are specified. The signatures $+/-$ indicate that the regions so marked are mapped 1 to 1 onto the upper/lower half plane. What is $f$? Explain why it cannot be otherwise.




**Problem.** (Jan 01 #1) Explain why the function $\sqrt{1-z^2}$ can be thought of as single valued in a plane cut along $-1 \leq z \leq 1$. Then the integral:

$$I = \int \frac{\mathrm{d}z}{\sqrt{1-z^2}}$$

taken around the circle $|z| = R > 1$ makes sense. How is $I$ related to $\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \frac{\pi}{2}$? How does $I$ change as $R \to \infty$? Compute $I$ by "pure thought" in light of these remarks.