Research Statement.

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1 Introduction.

In the early 70's Parshin introduced his notion of the multidimensional residue of a meromorphic top-form on an algebraic variety ([P]), which is a generalization of the classical one-dimensional residue. The main difference between Parshin's definition and the one-dimensional case is that in higher dimensions one computes the residue not at a point but at a complete flag of subvarieties $V_n \supset \cdots \supset V_0$, $\dim V_k = k$. Parshin and Lomadze also proved the Reciprocity Law for residues ([P], [L]): if one fixes all elements of the flag, except V_k, where $0 < k < n$, and considers all possible choices of V_k, then only finitely many of these choices give non-zero residues, and the sum of these residues is zero.

Parshin's constructions are purely algebraic. In fact, they work in very general settings, not only over complex numbers. However, in the complex case one would expect a more geometric variant of the theory.

Some work in this direction was done by J.-L. Brylinski and D. A. McLaughlin in [BM]. They introduced the *flag localized homology groups* and constructed a class in it, so that the residue is equal to the integral over this class. However, their constructions are not very explicit. In particular, there is no obvious way to extract a representative of the class from it.

We develop two other geometric approaches to the theory of Parshin residues. One uses the John Mather's theory of Abstract Stratified Spaces ([M]). Another uses the Resolution of Singularities techniques.

2 Residues via Coboundary Operators for Stratified Spaces.

The main results of the first part are as follows:

Let V be an abstract stratified space with oriented strata. The set of strata inherit the natural partial ordering. Namely, one says that X is *less* then Y if X belongs to the closure of Y. One says that X and Y are *consecutive strata* if X is less then Y and there is no strata in between them.

Let $X < Y$ be consecutive strata, let $\dim X = n$ and $\dim Y = k$. We define the coboundary operators $\phi_{X,Y} : H_*(X) \to H_{*+k-n-1}(Y)$ by analogy with the smooth case, when X is a submanifold in M and $Y = M \backslash X$.

The coboundary operators $\phi_{X,Y}$ satisfy the following relation:

Theorem 1. *Let $X < Y$ be two strata. Let $Z_1, \ldots, Z_k$ be all strata in between X and Y. Suppose that $Z_1, \ldots, Z_k$ are incomparable in pairs. Then ϕ_{X,Z_i} and $\phi_{Z_i,Y}$ are defined for all $i = 1, \ldots, k$, and*

$$\phi_{Z_1,Y} \circ \phi_{X,Z_1} + \phi_{Z_2,Y} \circ \phi_{X,Z_2} + \cdots + \phi_{Z_k,Y} \circ \phi_{X,Z_k} = 0.$$

One can express the Parshin residue in terms of coboundary operators as follows:

Let $F = \{V_n \supset \cdots \supset V_0\}$ be a flag of irreducible varieties. Fix any Whitney stratification of V_n, such that all elements of the flag are unions of strata and all strata are algebraic. For all k let $\check{V}_k \subset V_k$ be the stratum of top dimension in V_k.

Let ω be a meromorphic n-form on V_n. One can assume that ω is regular on $\check{V}_n$.

Theorem 2.
$$res_F(\omega) = \frac{1}{(2\pi i)^n} \int_{\Delta_n} \omega.$$
where $\Delta_n \in H_n(\check{V}_n)$ *is obtained from* $[\check{V}_0] \in H_0(\check{V}_0)$ *by applying the coboundary operators* $\phi_{\check{V}_k,\check{V}_{k+1}}$ *consecutively for* $k = 0, 1, \ldots, n-1$.

Note that the construction of coboundary operators is very geometric. In fact, it provides a smooth representative of the homology class Δ_n.

We also prove that for a given meromorphic form ω, there could be only finitely many non-zero residues and give a criteria how to find all of them:

Theorem 3. *Let* V *be an* n*-dimensional variety and* ω *a meromorphic* n*-form on* V*. Let* $\mathbf{S}_\omega$ *be any Whitney stratification of* V*, such that* ω *is regular on* $\check{V}$*. Let* $V_n \supset \cdots \supset V_0$ *be a flag of irreducible subvarieties. Suppose that at least one of* V_i*'s is not the closure of a stratum of* $\mathbf{S}_\omega$*. Then* $res_F\omega = 0$*.*

Parshin-Lomadze Reciprocity Law immediately follows from the Theorems 1, 2, and 3.

3 Residues via Resolution of Singularities.

In the second part we apply the resolution of singularities techniques to replace an arbitrary flag of varieties by a flag of smooth manifolds and an arbitrary meromorphic n-form by a locally almost-monomial one.

Let $F = \{V_n \supset \cdots \supset V_0\}$ be a flag of irreducible subvarieties. Let ω be a meromorphic n-form on V_n. Consider a resolution of singularities $\pi : \overline{V}_n \to V_n$, so that for any $k = 0, \ldots, n-1$, the preimage of V_k is a union of exceptional hypersurfaces, and the form $\pi^*(\omega)$ is regular on the complement to the exceptional divisor. We assume some additional geometric conditions on the resolution π, which are satisfied after some additional blow-ups with centers in intersections of exceptional hypersurfaces (see [M1] or [M2] for details).

Let $\overline{F} = \{\overline{V}_n \supset \cdots \supset \overline{V}_0\}$ be the flag of consecutive proper preimages of the flag F. It turns out, that the flag $\overline{F}$ has a very simple description in terms of the exceptional hypersurfaces:

Let D_k be the union of those exceptional hypersurfaces H, for which $\pi(H) = D_k$. We have the following Lemma:

Lemma 1. $\overline{V}_k = D_{n-1} \cap \cdots \cap D_k$ *and* $\overline{V}_k$ *is smooth for all* $k = 0, \ldots, n$.

Therefore, $\overline{F}$ is a natural smooth replacement of the flag F. Note, that the elements of the flag $\overline{F}$ are not irreducible any more. In particular, $\overline{V}_0$ is a finite set of points. However, number of points in $\overline{V}_0$ is independent on the resolution. In fact, we have the following Lemma:

Lemma 2. *For any $k = 0, \dots, n$, the birational type of $\overline{V}_k$ is independent on the choice of the resolution π.*

There is a way to express the Parshin residue in terms of very simple residues of the form $\pi^*(\omega)$:

$\overline{V}_0$ is a finite set of point. At each point of $\overline{V}_0$ exactly n exceptional hypersurfaces meet. Let $a \in \overline{V}_0$ and $(x_1, \dots, x_n)$ be local coordinates near a, such that the exceptional hypersurfaces coincide with the coordinate hyperplanes in a neighborhood of a.

Let $\gamma_a = \{|x_1| = |x_2| = \cdots = |x_n| = \epsilon\}$. Let $res_a \pi^*(\omega) = \frac{1}{(2\pi i)^n} \int\limits_{\gamma_a} \pi^*(\omega)$.

Theorem 4.

$$res_F(\omega) = \sum_{a \in \overline{V}_0} res_a \pi^*(\omega).$$

Remark. In fact, in the original Parshin's definition, the residue at a flag is defined as a sum of certain more delicate residues. We show, that the summands in the Theorem 4 are equal to the summands in the original Parshin's definition.

One can use the resolution of singularities to prove the Reciprocity Law:

Let $F = \{V_n \supset \cdots \supset \widehat{V_k} \supset \cdots \supset V_0\}$ be a non-complete flag of varieties. Let ω be a meromorphic n-form on V_n. Similarly to the above constructions, consider a resolution of singularities $\pi : \overline{V}_n \to V_n$, such that preimages of all V_i's are unions of strata, and $\pi^*(\omega)$ is regular on the complement to the exceptional divisor.

We will use the same notations as before: D_i denote the union of those exceptional hypersurfaces H, for which $\pi(H) = D_k$.

Consider the intersection $C = D_{n-1} \cap \cdots \cap D_{k+1} \cap D_{k-1} \cap \cdots \cap D_0$.

Lemma 3. *The irreducible components of C are smooth curves. The only singularities of C are simple intersections.*

Let $\Sigma \subset C$ be the set of points, where C intersects exceptional hypersurfaces, which are not subsets of D_i for any i. Similarly as before, there are exactly n exceptional hypersurfaces meeting at every point of $x \in \Sigma$. Therefore, one can define the residue $res_x(\pi^*\omega)$.

Theorem 5.

1. $\sum\limits_{x \in \Sigma} res_x(\pi^*\omega) = 0.$
2. *The Reciprocity Law for the form ω and the non-complete flag F is equivalent to the relation in the first part of the theorem.*

Remark. Relations similar to the first part of the Theorem 5 appeared in literature as the total sums of the Grothendieck residues over compact curves (see [T], for example). However, we did not manage to find a particular variation of the total sum theorem, which would imply the relation we need.

It is not true, that the sum of residues over those points of Σ, which belong to a particular irreducible component of C is zero. The problem is that there could be non-trivial residues at the points of self-intersection of C. As we sum over the whole C, the residues at the self-intersections are counted twice with opposite signs. However, in some situations the curve C naturally splits into several connected components. This implies, that the Reciprocity Law is, in fact, the sum of certain more delicate relation. This ramified version of the Reciprocity Law seems to be new, although we think that the arguments of A. Beilinson (see [B]) could be modified to cover this ramification.

All these results, except for the proof of the Reciprocity Law via resolution of singularities, are available in my PhD thesis [M1]. Accurate formulation of the results of the thesis can be found in [M2] (there are no proofs in this paper). The better source for the first part of the thesis is [M3]. I'm currently working on preparing the second part of the thesis for publication. I'm planning to include the proof of the Reciprocity Law via resolution of singularities in it. A preliminary version of the second part is available in [M4].

4 Research Plans.

There is an interesting observation: let $X \subset Y$ be algebraic varieties, such that X and $Y \backslash X$ are smooth. This does not imply, that $Y = X \sqcup (Y \backslash X)$ is a Whitney stratification. For example, take $Y = \{xy + z^2 = 0\} \subset \mathbb{C}^3$ and $X = \{x = z = 0\}$. However, one can subdivide X in such a way, that one gets a Whitney stratification.

Consider such a subdivision. Let $\check{X}$ and $\check{Y}$ be the open strata in X and Y correspondingly (in fact, $\check{Y} = Y \backslash X$).

Lemma 4. *The coboundary homomorphism* $\phi_{\check{X},\check{Y}} : H_*(\check{X}) \to H_{*+\dim Y - \dim X - 1}(\check{Y})$ *factors through the* $i_* : H_*(\check{X}) \to H_*(X)$*. In other words, there exist the homomorphism* $\phi_{X,\check{Y}} : H_*(X) \to H_{*+\dim Y - \dim X - 1}(\check{Y})$*, such that* $\phi_{\check{X},\check{Y}} = \phi_{X,\check{Y}} \circ i_*$.

The construction of the homomorphism $\phi_{X,\check{Y}}$ involve passing to the dual homomorphism in the Borel-Moore homology. We don't know how to construct it on the level of cycles.

This observation should imply an improved version of the Theorem 3. Namely, given a variety V_n and a meromorphic top-form ω on it, there should be a simple canonical way of constructing a subdivision of V_n into smooth pieces, such that all the non-trivial residues are captured by this subdivision. This subdivision does not need to be a Whitney stratification.

We plan to describe this subdivision and prove an improved version of the Theorem 3.

We also plan to use the resolution of singularities approach to investigate the relations between the Parshin Residues and the Grothendieck Residues.

References

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