

Geometric Theory of Parshin's Residues. Toric Neighborhoods of Parshin's Points.

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Abstract

The paper consist of two parts. In the first part we introduce flags of lattices and associated injective systems of (non-normal) cones and projective systems of (non-normal) affine toric varieties. We study the associated field of multidimensional Laurent power series. In the second part we use the resolution of singularities techniques to study the geometry near a complete flag of subvarieties and the Parshin's residues. The first part plays the role of a standard coordinate neighborhood for Parshin's points.

1 Flags of Lattices and the Associated Systems of Toric Varieties.

1.1 Ordered Abelian Groups.

Definition 1.1.1. An Abelian group G subdivided into subsets $G = G_- \sqcup \{0\} \sqcup G_+$ is called an *ordered abelian group* if

- (1) $a \in G_- \Rightarrow -a \in G_+$;
- (2) G_+ is a semigroup.

We write $a > b$ if $a - b \in G_+$ and $a < b$ if $a - b \in G_-$. Easy to see that this gives a total ordering of G . The elements of G_+ are called positive and the elements of G_- are called negative.

Definition 1.1.2. A subgroup $H \subset G$ is called *isolated* if for any two positive elements $g_1 > g_2 > 0$ such that $g_1 \in H$ it follows that $g_2 \in H$.

It follows immediately that if $h > 0$, $g > 0$, $h \in H$, and $g \notin H$ than $g > h$.

Theorem 1.1.1. Let $H^1 \subset G$ and $H^2 \subset G$ be two isolated subgroups. Then either $H_1 \subset H_2$ or $H_2 \subset H_1$.

Definition 1.1.3. Let $G = H^k \supset H^{k-1} \supset \dots \supset H^0 = \{0\}$ be the tower of all isolated subgroups of G . Then k is the *rank* of G .

Theorem 1.1.2. Suppose that G is isomorphic to \mathbb{Z}^n and has rank n . Let $G = H^n \supset H^{n-1} \supset \dots \supset H^0 = \{0\}$ be the isolated subgroups of G . Then for all k H^k is isomorphic to \mathbb{Z}^k and the order on G is isomorphic to the lexicographic order with respect to any basis (e_n, \dots, e_1) of G such that for all k (e_k, \dots, e_1) is a basis in H^k . (The lexicographic order range the elements first with respect to the coefficient at e_n , than e_{n-1} , etc.)

Definition 1.1.4. Let G be an ordered abelian group of rank greater than 1. Let $G' \subsetneq G$ be its maximal proper isolated subgroup. Than $H_+ := G_+ \setminus G'$ is called the *upper half-space* in G .

For groups of rank 1 we set $H_+ = G_+$ and for $G = \{0\}$ we set $H_+ = \{0\}$.

1.2 Flags of Lattices. Injective Systems of Cones and Projective Systems of Toric Varieties.

Let L^n be an ordered abelian group of rank n isomorphic to \mathbb{Z}^n . Let $L^n \supset \hat{L}^{n-1} \supset \dots \supset \hat{L}^1 \supset L^0$ be the isolated subgroups of L^n . Let $L^i \subset \hat{L}^i$ for $1 \leq i \leq n-1$ be subgroups of full rank such that $L^n \supset L^{n-1} \supset \dots \supset L^0$.

Definition 1.2.1. $L^n \supset L^{n-1} \supset \dots \supset L^0$ is called a *flag of lattices*.

We use the multiplicative notations for the operation in L^n .

Let H_+^k be the upper half-space in L^k for $k = 0, \dots, n$. We are interested in the semigroup $L = H_+^0 \cup \dots \cup H_+^n$ (i.e. we take 0, then all the positive elements of L^1 , then all the positive elements in L^2 which are not in \hat{L}^1 , etc.). Note, that L is not finitely generated. In particular, it doesn't correspond to any algebraic variety. However, one can consider L as a union of countably many cones, which are finitely generated.

Definition 1.2.2. Let $\hat{\mathcal{C}}$ be the set of all simple cones \hat{C} in L^n such that for all $k = 1, \dots, n$ exactly k generators of \hat{C} belong to \hat{L}_+^k (i.e. one of the generators is exactly the generator of \hat{L}_+^1 , another one belong to \hat{L}_+^2 , etc.).

Definition 1.2.3. Let $\hat{C} \in \hat{\mathcal{C}}$ and let (x_1, \dots, x_n) be the generators of \hat{C} such that $x_i \in \hat{L}_+^i$ for $i = 1, \dots, n$. Then we call (x_1, \dots, x_n) the *standard generators* of \hat{C} .

The cones from $\hat{\mathcal{C}}$ are not subsets of L . So we need to intersect them with L .

Definition 1.2.4. Let $\mathcal{C} = \{C = \hat{C} \cap L : \hat{C} \in \hat{\mathcal{C}}\}$.

Lemma 1.2.1. *Elements of \mathcal{C} are finitely generated.*

Proof: Let $C \in \mathcal{C}$ and let (x_1, \dots, x_n) be the standard generators of \hat{C} . Let k_1, \dots, k_n be the smallest positive integers such that $x_i^{k_i} \in C$. Denote $y_i := x_i^{k_i}$. Let $\hat{P} \subset \hat{C}$ be the subset in \hat{C} consisting of all the monomials $x_1^{d_1} \dots x_n^{d_n}$ such that $0 \leq d_i \leq k_i$ for $i = 1, \dots, n$. Let $P = \hat{P} \cap C$. P is obviously finite. Let us prove that it generates C .

Any element $x_1^{m_1} \dots x_n^{m_n} \in \hat{C}$ can be factored in the following way: $x_1^{m_1} \dots x_n^{m_n} = y_1^{l_1} \dots y_n^{l_n} x_1^{d_1} \dots x_n^{d_n}$, where $x_1^{d_1} \dots x_n^{d_n} \in \hat{P}$, $l_i \geq 0$, and $d_i \neq 0$ if and only if $m_i \neq 0$ for $i = 1, \dots, n$. Therefore, we only need to prove that if $x_1^{m_1} \dots x_n^{m_n} \in C$ then $x_1^{d_1} \dots x_n^{d_n} \in C$ as well.

Let $x_1^{m_1} \dots x_n^{m_n} \in C$. Let j be the biggest number such that $m_j \neq 0$, i.e. $x_1^{m_1} \dots x_n^{m_n} = x_1^{m_1} \dots x_n^{m_j}$ and $m_j \neq 0$. Then $d_n = \dots = d_{j+1} = 0$ and $d_j \neq 0$ as well. Since $x_1^{m_1} \dots x_n^{m_j} \in C$, it follows that $x_1^{m_1} \dots x_n^{m_j} \in L^j$ and $x_1^{m_1} \dots x_n^{m_j} \notin \hat{L}^{j-1}$. Since $y_1, \dots, y_j \in L^j$, $x_1^{d_1} \dots x_n^{d_n} \in L^j$ as well. Also, since $d_j \neq 0$, $x_1^{d_1} \dots x_n^{d_n} \notin \hat{L}^{j-1}$. So, $x_1^{d_1} \dots x_n^{d_n} \in L$ and, therefore, $x_1^{d_1} \dots x_n^{d_n} \in C$. \square

Lemma 1.2.2. *Let $K \subset L_+^n$ be a finite set of positive elements in L^n . Then there exist $\hat{C} \in \hat{\mathcal{C}}$ such that $K \subset \hat{C}$.*

Proof: It is enough to prove that if $\hat{C} \subset \hat{C}$ and $a \in L_+^n$ then there exist another cone $\hat{C}' \in \hat{\mathcal{C}}$ such that $\hat{C} \subset \hat{C}'$ and $a \in \hat{C}'$. Indeed, using this fact one can start from any cone in $\hat{\mathcal{C}}$ and add elements of K one by one.

Let (x_1, \dots, x_n) be the standard generators of \hat{C} . Let $a = x_1^{k_1} \dots x_n^{k_n}$. Let j be the biggest number such that $k_j \neq 0$. Note that $k_j > 0$, otherwise a is not positive. Take the cone \hat{C}' generated by $(x_1, \dots, x_{j-1}, x'_j, x_{j+1}, \dots, x_n)$, where $x'_j = x_1^{\min(k_1, 0)} \dots x_{j-1}^{\min(k_{j-1}, 0)} x_j$ (i.e. x'_j is obtained from $a = x_1^{k_1} \dots x_j^{k_j}$ by removing all the factors with positive powers and replacing $x_j^{k_j}$ by x_j).

$\hat{C}' \in \hat{\mathcal{C}}$ because $x'_j \in L_j$ and the transition matrix from (x_1, \dots, x_n) to $(x_1, \dots, x_{j-1}, x'_j, x_{j+1}, \dots, x_n)$ is integrally invertible. Indeed, $x_j = x_1^{-\min(k_1, 0)} \dots x_{j-1}^{-\min(k_{j-1}, 0)} x'_j$.

We need to check that $\hat{C} \subset \hat{C}'$ and $a \in \hat{C}'$. Indeed, all the generators of \hat{C} except for x_j are also generators of \hat{C}' and $x_j \in \hat{C}'$ because $x_j = x_1^{-\min(k_1, 0)} \dots x_{j-1}^{-\min(k_{j-1}, 0)} x'_j$ and $-\min(k_i, 0) \geq 0$ for $i = 1, \dots, j-1$. Also, $a = x_1^{k_1} \dots x_n^{k_n} = x_1^{k_1} \dots x_j^{k_j} = x_1^{k_1} \dots x_{j-1}^{k_{j-1}} (x_1^{-\min(k_1, 0)} \dots x_{j-1}^{-\min(k_{j-1}, 0)} x'_j)^{k_j} = x_1^{k_1 - k_j \min(k_1, 0)} \dots x_{j-1}^{k_{j-1} - k_j \min(k_{j-1}, 0)} (x'_j)^{k_j}$. Since $k_j > 0$, it follows, that $k_i - k_j \min(k_i, 0) \geq 0$. Therefore, $a \in \hat{C}'$. \square

Corollary 1. *Let $G \subset L_+^n$ be any finitely generated semigroup in L_+^n . Then there exist $\hat{C} \in \hat{\mathcal{C}}$ such that $G \subset \hat{C}$.*

Corollary 2. *Let $G \subset L$ be any finitely generated semigroup in L . Then there exist $C \in \mathcal{C}$ such that $G \subset C$.*

Corollary 3. $L = \bigcup_{C \in \mathcal{C}} C$.

Corollary 4. *Both $\hat{\mathcal{C}}$ and \mathcal{C} are injective systems under inclusion.*

Although the elements of \mathcal{C} are not cones in the usual sense, we still call them cones. If $C \in \mathcal{C}$ then the corresponding element $\hat{C} \in \hat{\mathcal{C}}$ is called the normalization of C .

There is the correspondence between the elements of \mathcal{C} and non-normal affine toric varieties T_C . (One can construct the variety T_C as follows: semigroup algebra of C is finitely generated and therefore correspond to the affine variety which we denote by T_C .) We denote the set of toric varieties T_C for all $C \in \mathcal{C}$ by \mathcal{T} .

Any $C \in \mathcal{C}$ has exactly one k -dimensional face which span L_k , for any $k = 0, \dots, n$. Therefore, one can identify one of the k -dimensional orbits in all $T \in \mathcal{T}$ for each $k = 0, \dots, n$. We denote these orbits T^0, T^1, \dots, T^n . So, $T^i \subset T$ for all $T \in \mathcal{T}$ and $i = 0, \dots, n$. Note also, that $T^k \cup T^{k-1}$ is an affine subvariety in every $T \in \mathcal{T}$ for $i = 1, \dots, n$. However, any union of more than two consequent orbits is not a subvariety.

Let $C, C' \in \mathcal{C}$ and $C' \subset C$. Then there is the natural map $\phi_{C, C'} : T_C \rightarrow T_{C'}$. Note, that $\phi_{C, C'}$ is identity on T^0, T^1, \dots, T^n . Since the maps $\phi_{C, C'}$ goes in the back direction, \mathcal{T} is a projective system with respect to these maps.

Take $T_C \in \mathcal{T}$. Although it is not normal, its normalization \widetilde{T}_C is isomorphic to \mathbb{C}^n . The standard generators (x_1, \dots, x_n) of the cone \widehat{C} give coordinates on \widetilde{T} . We'll use (x_1, \dots, x_n) as coordinates for T as well (although some points are going to be glued together). We call these coordinates the *standard coordinates* for the varieties $T \in \mathcal{T}$.

Switching from T_C to $T_{C'}$ corresponds to the change of coordinates

$$\begin{aligned} x'_1 &= x_1 \\ x'_2 &= x_1^{d_{12}} x_2 \\ &\vdots \\ x'_n &= x_1^{d_{1n}} \dots x_{n-1}^{d_{(n-1)n}} x_n, \end{aligned}$$

where

$$\begin{pmatrix} 1 & d_{12} & d_{13} & \dots & d_{1n} \\ 0 & 1 & d_{23} & \dots & d_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

is the transition matrix from the standard generators of the cone \widehat{C} to the standard generators of the cone \widehat{C}' .

The following lemma is important in the study of the Laurent series of L :

Lemma 1.2.3. *Let $\widehat{C}' \in \widehat{C}$ and $S \subset \widehat{C}'$, $S \neq \emptyset$. Then S contains its minimal element s_{min} and there exist another cone $\widehat{C} \in \widehat{C}$ such that $S \subset \widehat{C} + s_{min}$.*

Proof: Let (x_1, \dots, x_n) be the standard generators of \widehat{C}' . Note, that according to the Theorem 1.1.2 the order in L_n coincides with the lexicographic order with respect to the basis (x_1, \dots, x_n) (first, with respect to the power of x_n , then the power of x_{n-1} , etc.). Let m_n be the smallest integer such that $x_1^{k_1} \dots x_{n-1}^{k_{n-1}} x_n^{m_n} \in S$ for some integers k_1, \dots, k_{n-1} . Let m_{n-1} be the smallest integer such that $x_1^{k_1} \dots x_{n-2}^{k_{n-2}} x_{n-1}^{m_{n-1}} x_n^{m_n} \in S$ for some integers k_1, \dots, k_{n-2} . Continuing in the same way we get the integers m_1, \dots, m_n such that $s_{min} = x_1^{m_1} \dots x_n^{m_n} \in S$ is the minimal element in S .

Let $S' = \{a - s_{min} : a \in \widehat{C}' \text{ and } a > s_{min}\}$. Clearly, $S \subset S' + s_{min}$. Therefore, it is enough to find $\widehat{C} \in \widehat{C}$ such that $S' \subset \widehat{C}$.

Let $y_1 = x_1, y_2 = x_1^{-m_1} x_2, \dots, y_n = x_1^{-m_1} \dots x_{n-1}^{-m_{n-1}} x_n$. Easy to see that $y_k \in \widehat{L}_+^k$ for $k = 1, \dots, n$. Moreover, the transition matrix from (x_1, \dots, x_n) to (y_1, \dots, y_n) is upper triangular with units on diagonal. Therefore, the cone generated by (y_1, \dots, y_n) is simple and belong to \widehat{C} . Denote this cone by \widehat{C} . Let's prove that $S' \subset \widehat{C}$. Indeed, let $s = x_1^{l_1} \dots x_n^{l_n} \in S'$. It follows from the definition of S' , that $l_k \geq -m_k$ for $k = 1, \dots, n$ and that the last non-zero power, say l_p , is positive (otherwise $s < 0$). Then $s = y_p(x_1^{l_1+m_1} \dots x_{p-1}^{l_{p-1}+m_{p-1}} x_p^{l_p-1})$. Note, that all the powers in the last formula are non-negative and that $x_k \in \widehat{C}$ for all $k = 1, \dots, n$. Therefore, $s \in \widehat{C}$. \square

1.3 Laurent Power Series.

Definition 1.3.1. Let $F(L)^*$ be the set of all formal infinite linear combinations $f = \sum_{p \in L^n} f_p p$ of the elements of L^n such that

1. There exist at least one point in the torus T^n such that f is convergent at this point.
2. There exist a cone $\hat{C} \in \hat{\mathcal{C}}$ and an element $p_0 \in L^n$ such that the Newton polyhedron of f is a subset of $\hat{C} + p_0$.

Let $F(L) = F(L)^* \sqcup \{0\}$ (i.e. we add the series with zero coefficients to $F(L)^*$). We call $F(L)$ the set of Laurent series of L .

Lemma 1.3.1. *Let $f \in F(L)^*$. Then one can choose the cone \hat{C} and an element $p_0 \in L^n$ in such a way that the Newton polyhedron of f is a subset of $\hat{C} + p_0$ and $f_{p_0} \neq 0$.*

Proof: Let $Sup_f = \{p \in L^n : f_p \neq 0\}$. Since $f \in F(L)$, $Sup_f \subset \hat{C}' \in \hat{\mathcal{C}}$. Now we just apply the Lemma 1.2 to Sup_f to get p_0 and \hat{C} . \square

Note, that the $p_0 \in L^n$ from Lemma 1.3.1 is unique for each $f \in F(L)^*$. Basically, it is the smallest element in L^n such that the corresponding coefficient of f is not zero. Therefore, one gets a map $\nu : F(L)^* \rightarrow L^n$.

Lemma 1.3.2. *Let $f_1, \dots, f_m \in F(L)$. Then there exist $T \in \mathcal{T}$ such that $\frac{f_i}{\nu(f_i)}$ converge to holomorphic non-zero functions in a neighborhood of the origin in the normalization \tilde{T} of T for $i = 1, \dots, m$ (here (x_1, \dots, x_n) are the standard coordinates on T).*

Proof: Let $\hat{C}_i \in \hat{\mathcal{C}}$ be such that the Newton polyhedron of f_i is a subset of $\hat{C}_i + \nu(f_i)$. Let $\hat{C} \in \hat{\mathcal{C}}$ be such that $\hat{C}_i \subset \hat{C}$ for all $i = 1, \dots, m$ (the existence of \hat{C} follows from the Lemma 1.2.2). Let $C = \hat{C} \cap L$, $T = T_C$, and (x_1, \dots, x_n) be the standard coordinates in T . Then, for any $i = 1, \dots, m$, $\frac{f_i}{\nu(f_i)}$ is a Taylor series in (x_1, \dots, x_n) converging at least at one point in $T^n \subset T$ and, therefore, converging in a neighborhood of zero in \tilde{T} . Moreover, since the coefficient of f_i at $\nu(f_i)$ is not zero, f_i is not zero at the origin. \square

Remark. Note, that $\nu(f_i)$'s are monomials in (x_1, \dots, x_n) . So, f_1, \dots, f_m are *almost monomial* in the neighborhood of the origin in \tilde{T} , i.e. each of them is equal to a monomial multiplied by a non-zero holomorphic function.

Theorem 1.3.1. *$F(L)$ is a field and $\nu : F(L)^* \rightarrow L^n$ is a homomorphism of the multiplicative group $F(L)^*$ of $F(L)$ to the ordered abelian group L^n .*

Proof: The theorem follows immediately from the previous lemma. Indeed, if $f, g \in F^*(L)$ then there exist $T \in \mathcal{T}$ with coordinates (x_1, \dots, x_n) such that both $\frac{f}{\nu(f)}$ and

$\frac{g}{\nu(g)}$ converge to holomorphic non-zero functions in a neighborhood of the origin in \tilde{T} . Let $\nu(f) = x_1^{k_1} \dots x_n^{k_n}$ and $\nu(g) = x_1^{m_1} \dots x_n^{m_n}$. Then $\frac{fg}{\nu(f)\nu(g)}$, $\frac{f+g}{x_1^{\min(k_1, m_1)} \dots x_n^{\min(k_n, m_n)}}$, and $\frac{f^{-1}}{\nu(f)^{-1}}$ also converge to holomorphic functions in a neighborhood of the origin. Moreover, $\frac{fg}{\nu(f)\nu(g)}$ and $\frac{f^{-1}}{\nu(f)^{-1}}$ don't vanish at the origin. Therefore, $fg, f^{-1}, f+g \in F(L)$, $\nu(fg) = \nu(f)\nu(g)$, and $\nu(f^{-1}) = \nu(f)^{-1}$. \square

Definition 1.3.2. A homomorphism of the multiplicative group of a field F to an ordered abelian group G is called a valuation on F .

According to the Theorem 1.3.1, the field $F(L)$ is endowed with a valuation.

Lemma 1.3.3.

1. Let r be a meromorphic function on T^n (which is the same as a meromorphic function on any $T \in \mathcal{T}$). Then there exist a Laurent series $f \in F(L)$ normally converging to r in $\{0 < |x_1| < \epsilon_1, 0 < |x_2| < \epsilon_2, \dots, 0 < |x_n| < \epsilon_n\} \subset T^n$, where (x_1, \dots, x_n) are the standard coordinates on one of the varieties $T \in \mathcal{T}$.
2. Let $T \in \mathcal{T}$. Let r be a meromorphic function on a neighborhood of the origin in T . Then there exist a Laurent series $f \in F(L)$ normally converging to r in $\{0 < |x_1| < \epsilon_1, 0 < |x_2| < \epsilon_2, \dots, 0 < |x_n| < \epsilon_n\} \subset T^n$, where (x_1, \dots, x_n) are the standard coordinates on one of the varieties $T' \in \mathcal{T}$.

Proof: Both statements immediately follows from the Theorem 1.3.1. Indeed, the Taylor expansions of functions holomorphic at the origin in the normalization of any $T \in \mathcal{T}$ belong to $F(L)$ and $F(L)$ is closed under division by non-zero elements. \square

$F(L)$ is basically the field of functions, which are meromorphic in a "good" neighborhood of the flag T^0, T^1, \dots, T^n . Note, that $F(L)$ and the valuation on it depend only on L^n , and don't depend on other elements of the flag $L^0 \subset L^1 \subset \dots \subset L^n$. However, if one wants to consider the subalgebra in $F(L)$ of functions which are holomorphic in a "good" neighborhood of the flag T^0, T^1, \dots, T^n , it will depend on the whole flag $L^0 \subset L^1 \subset \dots \subset L^n$ (or, equivalently, on the semigroup L).

Definition 1.3.3. Let $O(L) \subset F(L)$ be the subset of all series $f = \sum_{p \in L^n} f_p p$ in $F(L)$ such that $f_p \neq 0 \Rightarrow p \in L$.

Remark. Let $f \in F(L)$ and $f = \nu(f)\phi(x_1, \dots, x_n)$ where (x_1, \dots, x_n) are the coordinates on the appropriate $T \in \mathcal{T}$ and ϕ is a holomorphic function in the neighborhood of the origin in the normalization $\tilde{T} \simeq \mathbb{C}^n$. Then the series of f belong to $O(L)$ if and only if $\nu(f) \geq 0$ and ϕ can be pushed forward to T .

Lemma 1.3.4. $O(L) \subset F(L)$ is a subalgebra.

Proof: Follows immediately from the definition. \square

Remark. Let $C \in \mathcal{C}$ be a cone. Denote by $O(C)$ the ring of series of elements of C , converging in a neighborhood of the origin in T_C . In other words, $O(C)$ is the ring of germs of analytic functions on T_C at the origin. Then $O(L) = \bigcup_{C \in \mathcal{C}} O(C)$.

Therefore, $O(L)$, in a sense, is the ring of germs of functions on the inverse limit of \mathcal{T} at the origin.

Lemma 1.3.5. *Any $T \in \mathcal{T}$ satisfy the following continuation property. Let $U \subset T$ be any open subset and $f : U \rightarrow \mathbb{C}$ be any continuous function, holomorphic in the complement to an analytic subset $\Sigma \subset U$ of codimension 1. Then f is holomorphic in U .*

Proof: It is enough to check that f is holomorphic in a neighborhood of every point in U . Moreover, every point in T has a neighborhood isomorphic to a neighborhood of the origin in a variety constructed in the same way as T , but for a different flag of lattices. Therefore, it is enough consider the case when U contains the origin and to check that f is holomorphic in a neighborhood of the origin.

Let (x_1, \dots, x_n) be the standard coordinates on T and $C \in \mathcal{C}$ be the corresponding cone. Let $\pi : \tilde{T} \rightarrow T$ be the normalization map. Then $\tilde{f} = f \circ \pi$ is continuous function in a neighborhood of the origin in $\tilde{T} \simeq \mathbb{C}^n$ and holomorphic in the complement to the analytic subset $\pi^{-1}(\Sigma)$ of codimension 1. Therefore, by Riemann Extension Theorem, it is holomorphic in this neighborhood and can be expanded into the Taylor series $\tilde{f} = \sum a_{\bar{k}} x^{\bar{k}}$ in it.

Since f is continuous in a neighborhood of the origin in T , it follows that if $a_{\bar{k}} \neq 0$ then $x^{\bar{k}} \in C$. Therefore, $\sum a_{\bar{k}} x^{\bar{k}} \in O(C)$ and f is regular at the origin in T . \square

1.4 Changes of Variables.

Let (f_1, \dots, f_n) be an n -tuple of functions from $F(L)$ and $x_1 = \nu(f_1), \dots, x_n = \nu(f_n)$. Suppose that:

1. (x_1, \dots, x_n) are the standard coordinates on a variety in \mathcal{T} ;
2. for any integers k_1, \dots, k_n such that $x_1^{k_1} \dots x_n^{k_n} \in L$ we have $f_1^{k_1} \dots f_n^{k_n} \in O(L)$.

Let L_f^n be the lattice of monomials in f_1, \dots, f_n . Easy to see that the restriction $\phi := \nu|_{L_f^n} : L_f^n \rightarrow L^n$ is an isomorphism. Consider the flag of lattices $L_f^0 \subset L_f^1 \subset \dots \subset L_f^n$, where $L_f^k = \phi^{-1}(L^k)$ for $k = 0, \dots, n$ and the order on L_f^n is induced by ϕ . Denote by L_f the corresponding semigroup, \mathcal{C}_f the corresponding system of cones and \mathcal{T}_f the corresponding system of toric varieties.

Theorem 1.4.1. *There exist a toric variety $T_f \in \mathcal{T}_f$ with the standard coordinates (g_1, \dots, g_n) , and a toric variety $T \in \mathcal{T}$ such that the n -tuple (g_1, \dots, g_n) provides an analytic isomorphism of a neighborhood of the origin in T to a neighborhood of the origin in T_f (note, that g_1, \dots, g_n are monomials in f_1, \dots, f_n).*

Proof: According to the Theorem 1.3.2 there exist a toric variety $T \in \mathcal{T}$ such that $\frac{f_k}{x_k}$ converge to a non-zero holomorphic function in a neighborhood of zero of the normalization \tilde{T} for $1 \leq k \leq n$. Let (y_1, \dots, y_n) be the standard coordinates in T and $C \in \mathcal{C}$ be the corresponding cone. The transition matrix from (x_1, \dots, x_n) to (y_1, \dots, y_n) is integer and upper-triangular with units on diagonal. For instance, let $y_k = x_1^{d_{1k}} \dots x_{k-1}^{d_{(k-1)k}} x_k$ for $k = 1, \dots, n$. Let $g_k = f_1^{d_{1k}} \dots f_{k-1}^{d_{(k-1)k}} f_k$ for $k = 1, \dots, n$. Let $\hat{C}_f \in \hat{\mathcal{C}}_f$ be the cone generated by (g_1, \dots, g_n) and let $T_f \in \mathcal{T}_f$ be the corresponding toric variety. Then $\nu(g_k) = y_k$ for $k = 1, \dots, n$ and $\frac{g_k}{y_k}$ converge to holomorphic functions in a neighborhood of the origin in the normalization \tilde{T} of T for $k = 1, \dots, n$. Let $h_k = \frac{g_k}{y_k}$. Then $\frac{\partial g_i}{\partial y_j}(0) = \frac{\partial h_i y_i}{\partial y_j}(0) = h_i(0) \frac{\partial y_i}{\partial y_j}(0) + \frac{\partial h_i}{\partial y_j}(0) y_i(0) = h_i(0) \delta_{i,j}$. Since $h_i(0) \neq 0$ for $i = 1, \dots, n$, the Inverse Function Theorem is applicable. Therefore, (g_1, \dots, g_n) provides an isomorphism $\hat{G} : \tilde{U} \rightarrow \tilde{U}_f$ of a polydisk \tilde{U} with center at the origin of \tilde{T} to a neighborhood of the origin \tilde{U}_f in \tilde{T}_f .

Let $\pi(\tilde{U}) = U \subset T$ and $\pi_f(\tilde{U}_f) = U_f \subset T_f$. We need to show, that the isomorphism $\hat{G} : \tilde{U} \rightarrow \tilde{U}_f$ can be pushed down to an isomorphism $G : U \rightarrow U_f$. According to the Lemma 1.3.5, it is enough to show that G is a homeomorphism. Moreover, since the topology of U (U_f respectively) is the factor topology of $\pi : \tilde{U} \rightarrow U$ ($\pi_f : \tilde{U}_f \rightarrow U_f$ respectively), it is enough to show, that G is a bijection.

First, we construct the map G and then prove that it is surjective and injective. We get that it is regular for free, but the above arguments allow us to avoid considering the inverse map. The condition 2 provides that the generators of C_f are regular (and, therefore, well defined) on U . Indeed, they are regular on \tilde{U} and they belong to $O(L)$ by condition 2. Surjectivity follows immediately from the diagram. All we need to prove is the injectivity.

Easy to see, that \hat{G} maps the coordinate cross to the coordinate cross and, in particular, maps $\tilde{T}^k \cap \tilde{U}$ to $\tilde{T}_f^k \cap \tilde{U}_f$, where \tilde{T}^k and \tilde{T}_f^k are respectively the coordinate subspaces $\tilde{T}^k = \{y_n = \dots = y_{k+1} = 0\} \subset \tilde{T}$ and $\tilde{T}_f^k = \{g_n = \dots = g_{k+1} = 0\} \subset \tilde{T}_f$. Therefore, for any point $x \in U$ the number of preimages of x in \tilde{U} is bigger or equal to the number of preimages of $G(x)$ in \tilde{U}_f . But the preimages of x are mapped injectively by \hat{G} to the preimages of $G(x)$. Therefore, $\hat{G}|_{\pi^{-1}(x)}$ is a bijection to $\pi_f^{-1}(G(x))$, which implies the injectivity of G . \square

Switching from \mathcal{T} to \mathcal{T}' is called a *change of variables*. Easy to see, that a change of variables gives an isomorphism from $F(L)$ to $F(L_f)$.

The same arguments works for any bigger cone $C' \supset C$. More precisely, we have the following

Corollary 1. *Let $C \in \mathcal{C}$ and $C_f \in \mathcal{C}_f$ be as in the Theorem 1.4.1. Let $C' \in \mathcal{C}$ be such that $C \subset C'$. Let $C'_f \in \mathcal{C}_f$ be such that the transition matrix from the standard generators of the \hat{C}'_f to the standard generators of the \hat{C}_f is the same as the transition matrix for the generators of \hat{C}' and \hat{C} respectively. Then there exist neighborhoods*

of the origins in $T_{C'}$ and $T_{C'_f}$, U' and U'_f and the unique isomorphism $G' : U' \rightarrow U'_f$ such that $G' \circ \phi_{C',C} = \phi_{C'_f,C'_f} \circ G'$ on U' .

So, the change of variables provides isomorphisms between the neighborhoods of the origins in the elements of \mathcal{T} and \mathcal{T}_f respectively, at least starting from some $T \in \mathcal{T}$ and $T_f \in \mathcal{T}_f$. These isomorphisms commute with the maps $\phi_{C',C} : T_{C'} \rightarrow T_{C'_f}$.

1.5 Residue.

One can consider the free one-dimensional module $\Omega(L)$ over $F(L)$ with the generator $\omega_{T^n} = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$. Note, that ω_{T^n} doesn't depend on the choice of coordinates (x_1, \dots, x_n) . Then for every element of $\omega \in \Omega(L)$ there exist a toric variety $T \in \mathcal{T}$ with coordinates (x_1, \dots, x_n) such that $\omega = x_1^{d_1} \dots x_n^{d_n} \phi dx_1 \wedge \cdots \wedge dx_n$ where ϕ is a holomorphic non-zero function in a neighborhood of the origin in \tilde{T} . We call the elements of $\Omega(L)$ the germs of meromorphic n -forms at the flag of orbits T^n, T^{n-1}, \dots, T^0 . According to the Lemma 1.3.3, any meromorphic form on a neighborhood of the origin in any $T \in \mathcal{T}$ can be expanded into a power series from $\Omega(L)$.

Definition 1.5.1. Let $\omega \in \Omega(L)$, $\omega = (\sum_{p \in L^n} a_p p) \omega_{T^n}$. Then the residue of ω is given by $res(\omega) = a_1$.

Lemma 1.5.1.

$$res(\omega) = \frac{1}{(2\pi i)^n} \int_{\tau^n} \omega,$$

where $\tau^n = \{(x_1, \dots, x_n) \in T^n : |x_1| = \epsilon_1, \dots, |x_n| = \epsilon_n\}$ for (x_1, \dots, x_n) – coordinates on a toric variety $T \in \mathcal{T}$, and ϵ_i are small enough, so that ω converges on τ^n . The orientation on τ^n is provided by the form $\frac{1}{(2\pi i)^n} \omega_{T^n}|_{\tau^n}$ (note, that this form is real).

Proof: Follows immediately from the Fubini's Theorem and the formula for the one-dimensional residues. \square

Suppose that the n -tuple of functions (f_1, \dots, f_n) defines a change of variables. Let $\omega \in \Omega(L)$ be a germ of a meromorphic form. Let $\hat{G} : \tilde{U} \rightarrow \tilde{U}_f$ be an isomorphism of neighborhoods of the origins in the normalizations of the appropriate toric varieties $T \in \mathcal{T}$ and $T_f \in \mathcal{T}_f$ provided by the change of variables (see Theorem 1.4.1). According to the Corollary 1 from the Theorem 1.4.1, one can assume that ω converges to a meromorphic form in $U \subset T$. Then one can push forward ω using the isomorphism \hat{G} . The result can be expanded as a power series from $\Omega(L_f)$. Therefore, we get a map from $F_* : \Omega(L) \rightarrow \Omega(L_f)$. Easy to see, that this map is an isomorphism of $F(L)$ -modules ($\Omega(L_f)$ is an $F(L)$ -module via the isomorphism of $F(L)$ and $F(L_f)$ provided by the change of variables).

Lemma 1.5.2. *Changes of variables doesn't change the residue, i.e. $res(F_*(\omega)) = res(\omega)$.*

Proof: Follows from the Lemma 1.5.1 and the observation that \hat{G} restricts to an isomorphism of $\tilde{U} \cap T^n$ and $\tilde{U}_f \cap T_f^n$ and this isomorphism is homotopic to the identity map (here we identify T^n and T_f^n with the standard tori via the standard coordinates on T and T_f). \square

1.6 Algebraic functions.

Let X be an analytic (algebraic) variety and let g_0, \dots, g_k be regular functions on X . Consider the equation $g_0 + g_1 t \cdots + g_k t^k = 0$. Let $U = X \setminus (\text{sing}(X) \cup \{g_k = 0\} \cup \Sigma)$, where $\Sigma = \{Dis = 0\}$, where Dis is the discriminant of the polynomial $g_0 + g_1 t \cdots + g_k t^k$. Then there exist the k -sheeted covering $p : W \rightarrow U$ and a regular function f on W , such that for every point $x \in U$ the values of f on the preimage $p^{-1}(x)$ are exactly the roots of the equation $g_0 + g_1 t \cdots + g_k t^k$. In such a situation, we say that f is an algebraic function on X . We say that Σ is the divisor of branching of f .

Lemma 1.6.1. *Let X be an analytic (algebraic) variety and let g_0, \dots, g_k be regular functions on X . Suppose that there is an open subset $U \subset X$, such that $X \setminus U$ is a finite union of subvarieties of codimension 1, and a holomorphic function f on U such that f satisfy the equation $g_0 + g_1 t \cdots + g_k t^k = 0$ on U . Then f can be continued to a meromorphic function on X .*

Proof: Consider the function $\tilde{f} = g_k f$. Easy to see that it is holomorphic on U and satisfy the integral equation $g_0 g_k^{k-1} + g_1 g_k^{k-2} t + \cdots + g_{k-1} t^{k-1} + t^k = 0$ on U . Let's prove that \tilde{f} is locally bounded on X , i.e. for any point $x \in X$ there exist a neighborhood V of x , such that \tilde{f} is bounded in $V \cap U$. Indeed, assume that it is not true. The coefficients $g_i g_k^{k-i-1}$ are regular on X , and, therefore, locally bounded. So, there exist a neighborhood V of x and a constant $M > 1$, such that $|g_i g_k^{k-i-1}| < M$ for $0 \leq i \leq k-1$. Since \tilde{f} is not bounded in $V \cap U$, there exist a point $y \in V \cap U$, such that $|\tilde{f}(y)| > kM$. Then $|\tilde{f}^k(y)| > kM |\tilde{f}^{k-1}(y)|$. But $|g_i(y) g_k^{k-i-1}(y) \tilde{f}^i(y)| < M |\tilde{f}^i(y)| \leq M |\tilde{f}^{k-1}(y)|$ for $0 \leq i \leq k-1$. So, $|g_0 g_k^{k-1} + g_1 g_k^{k-2} \tilde{f}(y) + \cdots + g_{k-1} \tilde{f}^{k-1}(y)| < |\tilde{f}^k(y)|$. Therefore, \tilde{f} doesn't satisfy the equation $g_0 g_k^{k-1} + g_1 g_k^{k-2} t + \cdots + g_{k-1} t^{k-1} + t^k = 0$ at y , which is a contradiction.

Let $p : \tilde{X} \rightarrow X$ be the normalization. Then $\tilde{f} \circ p$ is holomorphic in the complement of a finite union of subvarieties of codimension 1 in the normal variety \tilde{X} and is locally bounded in \tilde{X} . Therefore, it is regular on \tilde{X} . The normalization map p is a birational isomorphism, so \tilde{f} is meromorphic on X . Finally, $f = \frac{\tilde{f}}{g_k}$, therefore f is also meromorphic on X . \square

Let now $g_0, \dots, g_k \in F(L)$. One can choose $T \in \mathcal{T}$ such that g_k and the discriminant of the equation $g_k t^k + \cdots + g_0 = 0$ are almost monomial in a polydisk D with center at the origin of the normalization $\tilde{T} \simeq \mathbb{C}^n$. Let $X = D \cap T^n$. Let $p : W \rightarrow X$ be the corresponding k -sheeted covering. Suppose that W is connected (which is equivalent to say that the equation $g_k t^k + \cdots + g_0 = 0$ is irreducible). We need the following Lemma:

Lemma 1.6.2. *Let $N = k!$. Consider the map $P : \mathbb{C}^n \rightarrow \mathbb{C}^n$ given by $P(x_1, \dots, x_n) = (x_1^N, \dots, x_n^N)$. Let $W' = P^{-1}(X)$. Then there exist a map $\phi : W' \rightarrow W$ such that $p \circ \phi = P$.*

Proof: According to the classical theory, the connected coverings of X are classified up to isomorphism by the subgroups of the fundamental group $\pi_1(X)$ and the number of sheets corresponds to the index of the subgroup. In more details, the induced homomorphism $p_* : \pi_1(W) \rightarrow \pi_1(X)$ is injective and its isomorphism type defines the isomorphism type of the covering $p : W \rightarrow X$.

Since $\pi_1(X) \simeq \mathbb{Z}^n$, it follows, that $p_*(\pi_1(W)) \subset \pi_1(X)$ is a sublattice of full rank and index k . Let (a_1, \dots, a_n) be the basis of $\pi_1(X)$ corresponding to the loops going around coordinate hyperplanes in positive direction. Then $a_i^N \in p_*(\pi_1(W))$ for all i . Consider now the covering $P : W' \rightarrow X$. Easy to see, that $P_*(\pi_1(W')) \subset \pi_1(X)$ is generated by (a_1^N, \dots, a_n^N) . Therefore, $\pi_1(W') \subset \pi_1(W)$ (here we identified $\pi_1(W)$ and $\pi_1(W')$ with their images in $\pi_1(X)$). Therefore, there exist a covering $\phi : W'' \rightarrow W$ such that $\phi_*(\pi_1(W'')) = \pi_1(W') \subset \pi_1(W) \subset \pi_1(X)$. But then the composition $p \circ \phi : W'' \rightarrow X$ and $P : W' \rightarrow X$ correspond to the same subgroup in $\pi_1(X)$ and, therefore, isomorphic. \square

According to the Lemmas 1.6.2 and 1.6.1, the equation $g_k t^k + \dots + g_0 = 0$ has a meromorphic solution f in $P^{-1}(D)$. Moreover, f is holomorphic in $W' = P^{-1}(D) \cap (\mathbb{C}^*)^n$. Therefore, we have the following theorem:

Theorem 1.6.1. *Let $g_0, \dots, g_k \in F(L)$ be such that the equation $g_k t^k + \dots + g_0 = 0$ is irreducible. Let $N = k!$. Then there exist a toric variety $T \in \mathcal{T}$ with the coordinates (x_1, \dots, x_n) , and the integers m_1, \dots, m_n such that the root f of the equation $g_k t^k + \dots + g_0 = 0$ can be written in the form of the Piezo series $f = \sqrt[N]{x_1}^{m_1} \dots \sqrt[N]{x_n}^{m_n} \sum_{i_1, \dots, i_n \geq 0} f_{i_1, \dots, i_n} \sqrt[N]{x_1}^{i_1} \dots \sqrt[N]{x_n}^{i_n}$ converging in a neighborhood of the origin in the normalization \tilde{T} .*

2 Toric Neighborhoods of Parshin's points.

2.1 Branched coverings and Generic components of the preimage of a hypersurface.

Definition 2.1.1. Let X and Y be non-empty pure-dimensional algebraic (analytic) varieties of the same dimension. An algebraic (analytic) map $f : Y \rightarrow X$ is called a *branched covering* if it is proper, surjective, and is a local isomorphism at a generic point (i.e. there is an open dense subset $U \subset Y$ consisting of smooth points of Y such that for every point $y \in U$ $f(y)$ is a smooth point of X and the differential of f has full rang at y).

Note, that a composition of two branched coverings is again a branched covering.

Definition 2.1.2. Let $f : Y \rightarrow X$ be a branched covering. Let $H \in X$ be a hypersurface in X . Then the *generic component of the preimage* (shortly, the *generic preimage*) $H_f \subset Y$ is the union of those irreducible components \hat{H}_i of the full preimage $f^{-1}(H)$ for which the restriction $f|_{\hat{H}_i} : \hat{H}_i \rightarrow H$ is a local isomorphism at a generic point.

Note, that if $f : Y \rightarrow X$ is a blow-up with a smooth center $C \subset X$, then the generic preimage H_f coincide with the strict transform of H unless C and H has common irreducible components.

Let $f : Y \rightarrow X$ and $g : Z \rightarrow Y$ be branched coverings. Let $H \subset X$ be a hypersurface. Then, clearly, $(H_f)_g = H_{f \circ g}$. Indeed, the restriction of $f \circ g$ to an irreducible component of $g^{-1}(f^{-1}(H))$ is a local isomorphism at a generic point if and only if g is a local isomorphism at a generic point of this component and f is a local isomorphism at a generic point of the image of this component.

Lemma 2.1.1. $f|_{H_f} : H_f \rightarrow H$ is a branched covering.

Proof: By Sard's Lemma, the image of those components of $f^{-1}(H)$ which are not components of H_f has measure 0 in H . Therefore, H_f is non-empty and its image is dense in H . Then, since H_f is a closed subset in Y , $f|_{H_f}$ is proper. So, we only need to prove that $f|_{H_f}$ is surjective to H . Indeed, let $x \in H$. Let $K \subset H$ be a compact neighborhood of x in H . Since $f|_{H_f}$ is proper, $N = f^{-1}(K) \cap H_f$ is compact. So, $f(N)$ is compact and, therefore, closed. Since $f(H_f)$ is dense in H , there exist a sequence $\{y_m\} \subset H_f$, such that $f(y_m) \rightarrow x$ and $\{f(y_m)\} \in K$. Then $\{y_m\} \subset N$ and $\{f(y_k)\} \subset f(N)$. Therefore, $x \in f(N)$. \square

Note, that the normalization map is always a branched covering.

Theorem 2.1.1. Let X be a pure-dimensional analytic (algebraic) variety. Let $\pi : Y \rightarrow X$ be a degree one branched covering and let \tilde{Y} be normal. Let $p : \tilde{X} \rightarrow X$ be the normalization. Let $H \subset X$ be a hypersurface. Then there is a natural map $\tilde{\pi} : H_\pi \rightarrow H_p$ which is a degree one branched covering.

Proof: According to the universal properties of the normalization, the map $\pi : Y \rightarrow X$ factors through the normalization, i.e. there exist a map $\tilde{\pi} : Y \rightarrow \tilde{X}$ such that $\pi = p \circ \tilde{\pi}$. Moreover, $(H_p)_{\tilde{\pi}} = H_\pi$. Therefore, it is enough to consider the case when X is normal. So, $\tilde{X} = X$, $H_p = H$, $\tilde{\pi} = \pi$, and we need to prove that $\pi|_{H_\pi} : H_\pi \rightarrow H$ is a degree one branched covering.

Let $\text{sing}(X) \subset X$ and $\text{sing}(Y) \subset Y$ be the singular loci of X and Y correspondingly. Since π is a proper map and $\text{sing}(Y)$ is a closed subvariety, it follows that $\pi(\text{sing}(Y)) \subset X$ is a closed subvariety in X . Since X and Y are normal, the codimensions of $\text{sing}(X)$, $\text{sing}(Y)$, and, therefore, $\pi(\text{sing}(Y))$ are at least 2. Let $X' = X \setminus (\text{sing}(X) \cup \pi(\text{sing}(Y)))$. Let $H' = H \cap X'$. Note, that H' is a complement to a closed subvariety of codimension at least 1 in H . Let $Y' = \pi^{-1}(X')$. Now X' and Y' are smooth, $\pi' := \pi|_{Y'} : Y' \rightarrow X'$ is still a degree one branched covering, and $H' \subset X'$ is a

hypersurface. Note, that $H'_{\pi'} = H_{\pi} \cap Y'$, and it is a complement to a subvariety of codimension at least 1 in H_{π} (although $Y \setminus Y'$ can have codimension 1).

Let $\text{crit}(\pi') \subset Y$ be the critical locus of π . Let $\tilde{X} = X' \setminus \pi'(\text{crit}(\pi'))$. Let $\tilde{Y} = \pi'^{-1}(\tilde{X})$. Then $\tilde{\pi} = \pi|_{\tilde{Y}} : \tilde{Y} \rightarrow \tilde{X}$ is an isomorphism. Indeed, it is non-degenerate and, therefore, unbranched covering of degree 1.

We need the following

Lemma 2.1.2. *Let M and N be analytic manifolds and $f : M \rightarrow N$ be a degree one branched covering. Then $\text{codim}(f(\text{crit}(f))) \geq 2$.*

Proof: Suppose that $\text{codim}(f(\text{crit}(f))) = 1$. Then there exist $p \in \text{crit}(f)$ such that

1. p is a smooth point of the hypersurface $\text{crit}(f)$;
2. $f^{-1}(f(\text{crit}(f)))$ coincide with $\text{crit}(f)$ in a neighborhood of p ;
3. $f(p)$ is a smooth point of the codimension 1 irreducible component of $f(\text{crit}(f))$;
4. $f|_{\text{crit}(f)}$ is a local isomorphism at p .

Therefore, there exist coordinate systems (x_1, \dots, x_n) and (y_1, \dots, y_n) respectively in a neighborhood U of p in M and in a neighborhood V of $f(p)$ in N , such that

1. $f(U) \subset V$.
2. $U \cap \text{crit}(f) = \{x_n = 0\}$;
3. $V \cap \text{crit}(f) = \{y_n = 0\}$;
4. $f(x_1, \dots, x_{n-1}, 0) = (x_1, \dots, x_{n-1}, 0)$.

So, the map $f|_U$ is given by

$$\begin{aligned} y_1 &= x_1 + x_n \phi_1(x); \\ &\vdots \\ y_{n-1} &= x_{n-1} + x_n \phi_{n-1}(x); \\ y_n &= x_n^k \phi_n(x), \end{aligned}$$

where $k \geq 1$ and ϕ_n is not divisible by x_n . Note, that $\phi_n(p) \neq 0$. Indeed, otherwise $\{\phi_n = 0\} \subset f^{-1}(f(\text{crit}(f))) = \{x_n = 0\}$ and ϕ_n is not divisible by x_n .

One can get rid of all the ϕ_i 's simply by changing the coordinates in a neighborhood of $p \in M$. Indeed, let

$$\begin{aligned} t_1 &= x_1 + x_n \phi_1(x); \\ &\vdots \\ t_{n-1} &= x_{n-1} + x_n \phi_{n-1}(x); \\ t_n &= x_n \sqrt[k]{\phi_n(x)}. \end{aligned}$$

Easy to check that the Jacobian is not zero, so (t_1, \dots, t_n) are indeed coordinates in a neighborhood of p . In t 's coordinates the map f is given by

$$\begin{aligned} y_1 &= t_1; \\ &\vdots \\ y_{n-1} &= t_{n-1}; \\ y_n &= t_n^k. \end{aligned}$$

Since a general point in N should have only one preimage, k should be equal to 1. But then f is non-degenerate at p . \square

So, the codimension of $\pi'(\text{crit}(\pi'))$ is at least 2. Therefore, $\tilde{H} := H' \cap \tilde{X}$ is a complement to a subvariety of degree at least 1 in H' . Its' preimage $\tilde{\pi}^{-1}(\tilde{H}) = \tilde{H}_{\tilde{\pi}}$ is also a complement to a subvariety of codimension at least 1 in H'_{π} . And $\pi|_{\tilde{H}} : \tilde{H}_{\tilde{\pi}} \rightarrow \tilde{H}$ is an isomorphism. So, $\pi|_{H_{\pi}} : H_{\pi} \rightarrow H$ is a degree one branched covering. \square

2.2 Resolution of Singularities for Flags.

To avoid difficulties with the resolution of singularities, we need to assume some compactness condition on the analytic varieties. For simplicity, let us assume that all the analytic varieties are restrictions of bigger analytic varieties to relatively compact open subsets.

We need the following Theorem which is a direct corollary of the famous Hironaka Theorem (...) on resolution of singularities:

Theorem 2.2.1. *Let X be a variety. Let Y_1, \dots, Y_K be closed subvarieties in X . Then there exist a branched covering of degree one $\pi : \tilde{X} \rightarrow X$ such that:*

1. \tilde{X} is smooth;
2. $\pi|_{\tilde{X} \setminus D}$ is an isomorphism to $\text{reg}(X) \setminus (Y_1 \cup \dots \cup Y_k)$, where $D = H_1 \cup \dots \cup H_N$ is a union of smooth exceptional hypersurfaces H_i , which simultaneously have only normal crossings. We denote $\mathcal{D} = \{H_1, \dots, H_N\}$ the set of exceptional hypersurfaces (let us always assume the exceptional hypersurfaces irreducible).
3. For any $k = 1, \dots, K$ $\pi^{-1}(Y_k)$ is a union of hypersurfaces from \mathcal{D} ;

In order to improve the resolution, we'll need to do additional blow-ups with centers in intersections of exceptional hypersurfaces. We'll need some simple properties of this type of blow-ups:

Lemma 2.2.1. *Let \tilde{X} , $D = H_1 \cup \dots \cup H_N$, and $\mathcal{D} = \{H_1, \dots, H_N\}$ be as in Theorem 2.2.1. Let $C = H_{i_1} \cap \dots \cap H_{i_k}$ and $\pi_C : \tilde{X}_C \rightarrow \tilde{X}$ be the blow-up with center in C . Let $H_C = \pi_C^{-1}(C)$ and $\tilde{H}_i \subset \tilde{X}_C$ be the strict transform of $H_i \in \mathcal{D}$. Denote $D_C := \tilde{H}_1 \cup \dots \cup \tilde{H}_N \cup H_C$ and $\mathcal{D}_C = \{\tilde{H}_1, \dots, \tilde{H}_N, H_C\}$. Then*

1. D_C again has only normal crossing;

2. $\tilde{H}_{i_1} \cap \cdots \cap \tilde{H}_{i_k} = \emptyset$;
3. $\pi_C|_{\tilde{H}_{i_1} \cap \cdots \cap \tilde{H}_{i_{k-1}}}$ is an isomorphism to $H_{i_1} \cap \cdots \cap H_{i_{k-1}}$;
4. if $C \not\subset H_j \in \mathcal{D}$ (i.e. $j \neq i_1, \dots, i_k$) then $\pi_C^{-1}(H_j) = \tilde{H}_j$.

Proof:

1. is a standard property of blow-ups;
2. is trivial;
3. follows immediately from the standard fact that if $W \supset U \supset V$ are smooth manifolds, $\pi_V : W_V \rightarrow W$ is the blow-up with center in V and U_V is the strict transform of U , then $\pi_V|_{U_V} : U_V \rightarrow U$ is the blow-up with center in V . Indeed, let $W = \tilde{X}$, $U = H_{i_1} \cap \cdots \cap H_{i_{k-1}}$, and $V = C$. Then $U_V = H_{i_1} \cap \cdots \cap H_{i_{k-1}}$ and V is a hypersurface in U , so $\pi_V|_{U_V}$ is an isomorphism to U ;
4. it follows that C is transversal to H_j because H_j is a hypersurface. In particular, the normal bundle of $H_j \cap C$ in H_j is canonically isomorphic to the normal bundle to C in \tilde{X} restricted to $H_j \cap C$. Therefore, the restriction $\pi_C|_{\pi_C^{-1}(H_j)}$ is the blow-up of H_j with center in $H_j \cup C$.

□

We'll do the additional blow-ups in $n - 1$ steps. We call the consequent strict transforms of the original hypersurfaces H_1, \dots, H_N the 0-type hypersurfaces and the consequent strict transforms of the new hypersurfaces appearing after k th step — the k -type hypersurfaces.

On the k th step we blow-up the intersections of $n - k + 1$ 0-type hypersurfaces. Note, that after the $(k - 1)$ th step the 0-type hypersurfaces cannot meet by more than $n - k + 1$ at one point. Therefore, the centers for the blow-ups on the k th step are disjoint and one can blow them up simultaneously. Also, the k -type hypersurfaces are always disjoint for $k > 0$ and the 0-type hypersurfaces are disjoint after $(n - 1)$ th step. It is convenient to label the k -type hypersurfaces by $(n - k + 1)$ -tuples of 0-type hypersurfaces.

Applying the above procedure, one get the following Lemma:

Lemma 2.2.2. *In the Theorem 2.2.1, one can assume also that \mathcal{D} satisfy the following conditions:*

1. Let $H_i, H_j \in \mathcal{D}$ and $H_i \cap H_j \neq \emptyset$. Then either $\pi(H_i) \subset \pi(H_j)$ or $\pi(H_i) \supset \pi(H_j)$;
2. Let $H_{i_1}, \dots, H_{i_k} \in \mathcal{D}$, $C := H_{i_1} \cap \cdots \cap H_{i_k} \neq \emptyset$, and $\pi(H_{i_1}) \subset \cdots \subset \pi(H_{i_k})$. Then for any irreducible component C^0 of C $\pi(C^0) = \pi(H_1)$.

Proof: Follows immediately from the Lemma 2.2.1.

□

As an immediate corollary of the Lemma 2.2.2, we get the following result on resolutions for flags of subvarieties.

Definition 2.2.1. Let (x_1, \dots, x_n) be a coordinate system in U (i.e. (x_1, \dots, x_n) maps U isomorphically to an open neighborhood of the origin in \mathbb{C}^n). A meromorphic function f is called *almost monomial* in U with respect to (x_1, \dots, x_n) if $f = x_1^{d_1} \dots x_n^{d_n} \phi$, where d_1, \dots, d_n are integers and ϕ is a holomorphic non-zero function on U .

Theorem 2.2.2. Let $V_n \supset V_{n-1} \supset \dots \supset V_0$ be a flag of algebraic (closed analytic) subvarieties with $\dim V_i = i$. Let f_1, \dots, f_k be meromorphic functions on V_n . Then there exist a flag of smooth algebraic (closed analytic) subvarieties $\bar{V}_n \supset \bar{V}_{n-1} \supset \dots \supset \bar{V}_0$ and a map $\pi : \bar{V}_n \rightarrow V_n$, such that

1. π is a degree one branched covering (in fact, a composition of blow-ups);
2. \bar{V}_k is the generic component of the preimage of V_k with respect to $\pi|_{\bar{V}_{k+1}}$ for $k = n-1, n-2, \dots, 0$;
3. for any point $a \in \bar{V}_i$ there exist a system of coordinates (called good coordinates) (x_1, \dots, x_n) in a neighborhood U of a in \bar{V}_n such that
 - (a) $\bar{V}_j \cap U = \{b \in U : x_n(a) = \dots = x_{j+1}(a) = 0\}$ for $j = i, i+1, \dots, n$;
 - (b) $\pi^* f_1, \dots, \pi^* f_k$ are almost monomial in U with respect to (x_1, \dots, x_n) .

Proof: We apply the Theorem 2.2.1 and Lemma 2.2.2 to V_n with subvarieties V_{n-1}, \dots, V_0 and all the divisors of f_1, \dots, f_k . Let $\pi : \bar{V}_n \rightarrow V_n$ be the resulting resolution map and $\mathcal{D} = \{H_1, \dots, H_N\}$ be the exceptional hypersurfaces. Denote $\mathcal{D}_k = \{H_i \in \mathcal{D} : \pi(H_i) = V_k\}$ and $D_k = \bigcup_{H_i \in \mathcal{D}_k} H_i$. Let $\bar{V}_n \supset \bar{V}_{n-1} \supset \dots \supset \bar{V}_0$ be the flag of consequent generic preimages.

Lemma 2.2.3. $\bar{V}_k = D_{n-1} \cap \dots \cap D_k$ and \bar{V}_k is smooth for all $k = 0, \dots, n$.

Proof: We first prove that $\bar{V}_k = D_{n-1} \cap \dots \cap D_k$ by induction starting from $\bar{V}_{n-1} = D_{n-1}$, which is trivial by dimension.

Suppose that we already proved $\bar{V}_{k+1} = D_{n-1} \cap \dots \cap D_{k+1}$. The inclusion $D_{n-1} \cap \dots \cap D_k \subset \bar{V}_k$ follows from the condition 2 of the Lemma 2.2.2 and the dimension counting. Indeed, $\dim(D_{n-1} \cap \dots \cap D_k) = n - ((n-1) - (k-1)) = k = \dim(V_k)$ and $\pi(D_{n-1} \cap \dots \cap D_k) = \pi(D_k)$ (unless $D_{n-1} \cap \dots \cap D_k = \emptyset$ in which case the inclusion is trivial). Therefore, every irreducible component of $D_{n-1} \cap \dots \cap D_k$ is mapped surjectively to V_k , and, by dimension, $\pi|_{D_{n-1} \cap \dots \cap D_k}$ is a local isomorphism at a general point.

Suppose now that $X \subset D_{n-1} \cap \dots \cap D_{k+1}$ is an irreducible subvariety of codimension 1 such that $\pi(X) = V_k$. Since $\pi^{-1}(V_k)$ is a union of hypersurfaces from \mathcal{D} , it follows that $X \subset H \in \mathcal{D}$, such that $\pi(H) \supset V_k$. Moreover, there exist $H_{n-1} \in \mathcal{D}_{n-1}, \dots, H_{k+1} \in \mathcal{D}_{k+1}$ such that $X \subset H_{n-1} \cap \dots \cap H_{k+1} \cap H$. Consider $\pi(H)$. It is

an irreducible subvariety in V_n such that it either contains or is contained in V_i for every $i = n - 1, \dots, k + 1$ and it contains V_k . Therefore, it coincide with one of the V_i 's for $i = n - 1, \dots, k$. We need to prove that $\pi(H) = V_k$. Suppose it is wrong and $\pi(H) = V_i$ for $i > k$. Then $\pi(H_{n-1} \cap \dots \cap H_{k+1} \cap H) = V_{k+1}$ which is impossible by dimension.

The only way $D_{n-1} \cap \dots \cap D_k$ can be singular is if two of its irreducible components intersect each other. That means that there are two different sets of hypersurfaces H_{n-1}, \dots, H_k and H'_{n-1}, \dots, H'_k with $H_i, H'_i \in \mathcal{D}_i$, such that $C := H_{n-1} \cap \dots \cap H_k \cap H'_{n-1} \cap \dots \cap H'_k \neq \emptyset$. Since there is at least $n - k + 1$ different hypersurfaces, $\dim(C) < k$. However, by the condition 2 of the Lemma 2.2.2, $\pi(C) = V_k$ which is impossible by dimension. \square

The Theorem 2.2.2 is proved. \square

Definition 2.2.2. The flag $\bar{V}_n \supset \bar{V}_{n-1} \supset \dots \supset \bar{V}_0$ together with the map $\pi : \bar{V}_n \rightarrow V_n$ is called the *resolution of singularities of the flag* $V_n \supset V_{n-1} \supset \dots \supset V_0$, respecting the functions f_1, \dots, f_k .

2.3 Parshin's point, local parameters, and residue.

Let V_n be an algebraic (analytic) variety of dimension n . Let $V_n \supset \dots \supset V_0$ be a flag of subvarieties of dimension $\dim V_k = k$ in V_n .

Consider the following diagram:

$$\begin{array}{c}
 V_n \supset V_{n-1} \supset \dots \supset V_1 \supset V_0 \\
 \uparrow p_n \quad \uparrow p_n \\
 \tilde{V}_n \supset W_{n-1} \\
 \uparrow p_{n-1} \\
 \tilde{W}_{n-1} \supset \dots
 \end{array} \tag{1}$$

$$\begin{array}{c}
 \dots \supset W_1 \\
 \uparrow p_1 \\
 \tilde{W}_1 \supset W_0
 \end{array}$$

where

1. $p_n : \tilde{V}_n \rightarrow V_n$ is the normalization;
2. $W_{n-1} \subset \tilde{V}_n$ is the general preimage of V_{n-1} under p_n ;
3. for every $k = 1, 2, \dots, n - 1$

- (a) $p_k : \widetilde{W}_k \rightarrow W_k$ is the normalization;
- (b) $W_{k-1} \subset \widetilde{W}_k$ is the general preimage of V_{k-1} under $p_n \circ \cdots \circ p_k$.

Definition 2.3.1. We call the diagram 1 the *normalization diagram* of the flag $V_n \supset \cdots \supset V_0$.

Definition 2.3.2. The flag $V_n \supset \cdots \supset V_0$ of irreducible subvarieties together with a choice of a point $a_\alpha \in W_0$ is called a *Parshin's point*.

Remark. Consider $\{V_n \supset \cdots \supset V_0, a_\alpha \in W_0\}$ where V_i 's are not necessary irreducible. The choice of a_α provides the irreducible components of V_i 's. Therefore, $\{V_n \supset \cdots \supset V_0, a_\alpha \in W_0\}$ defines a Parshin's point $\{V'_n \supset \cdots \supset V'_0, a_\alpha \in W'_0\}$, where V'_i are the corresponding irreducible components and $W'_0 \subset W_0$. (Note that the diagram (1) for the irreducible flag $V'_n \supset \cdots \supset V'_0$ consists of irreducible components of the elements of the diagram (1) for the original flag $V_n \supset \cdots \supset V_0$.) However, if we wouldn't require irreducibility of V_i 's, $\{V_n \supset \cdots \supset V_0, a_\alpha\}$ and $\{V'_n \supset \cdots \supset V'_0, a_\alpha\}$ would formally be different Parshin's points, which is wrong. Sometimes we'll say Parshin's point $\{V_n \supset \cdots \supset V_0, a_\alpha\}$ without assuming that V_i 's are irreducible, by using the convention $\{V_n \supset \cdots \supset V_0, a_\alpha\} = \{V'_n \supset \cdots \supset V'_0, a_\alpha\}$.

$W_{i-1} \subset \widetilde{W}_i$ is a hypersurface in a normal variety. It follows that there exist a (meromorphic) function u_i on \widetilde{W}_i which has zero of order 1 at a generic point of W_{i-1} . Since meromorphic functions are the same on W_i and \widetilde{W}_i , one can consider u_i as a function on W_i . Then one can continue u_i to \widetilde{W}_{i+1} and so on. For simplicity, we denote all these functions by u_i . Now u_i is defined on V_n , and W_j and \widetilde{W}_j for $j \geq i$.

Definition 2.3.3. Functions (u_1, \dots, u_n) are called *local parameters*.

Let ω be a meromorphic n -form on V_n . Since the differentials du_1, \dots, du_n are linearly independent at a generic point of V_n , one can write

$$\omega = f du_1 \wedge \cdots \wedge du_n,$$

where f is a meromorphic function on V_n .

Furthermore, f can be expand into a power series in u_n with coefficients in meromorphic functions on W_{n-1} . Let f_{-1} be the coefficient at u_n^{-1} in this expansion. Then $\omega_{n-1} = f_{-1} du_1 \wedge \cdots \wedge du_{n-1}$ is a meromorphic form on W_{n-1} . Continuing in the same way, we get a function ω_0 on the finite set W_0 .

Definition 2.3.4. The residue of ω at the Parshin's point $P = \{V_n \supset \cdots \supset V_0, a_\alpha \in W_0\}$ is $res_P(\omega) := \omega_0(a_\alpha)$.

Definition 2.4.2. A set (y_1, \dots, y_n) of almost monomial functions in a neighborhood of $b_\alpha \in \overline{V}_n$ with respect to the coordinates (x_1, \dots, x_n) , such that the corresponding valuation matrix D is lower-diagonal with units on diagonal is called *generalized local parameters*.

Lemma 2.4.2 basically says that if (u_1, \dots, u_n) are local parameters and the resolution $\pi : \overline{V}_n \rightarrow V_n$ is such that $(v_1, \dots, v_n) := (u_1 \circ \pi, \dots, u_n \circ \pi)$ are almost monomial near b_α then (v_1, \dots, v_n) are generalized local parameters. However, later we'll have to use generalized local parameters of more general type.

We have the following simple lemma about the almost monomial functions:

Lemma 2.4.3. *Let (y_1, \dots, y_n) be almost monomial functions with respect to the good coordinates (x_1, \dots, x_n) in a neighborhood of b_α . Let $D = \{d_{ij}\}$ be the valuation matrix. Let $C = \{c_{ij}\} = D^{-1}$, and let*

$$\begin{aligned} z_1 &= y_1^{c_{11}} \dots y_1^{c_{1n}}; \\ z_2 &= y_1^{c_{21}} \dots y_1^{c_{2n}}; \\ &\vdots \\ z_n &= y_1^{c_{n1}} \dots y_n^{c_{nn}}. \end{aligned}$$

Then (z_1, \dots, z_n) are good coordinates in a neighborhood of b_α as well. (If $|\det D| \neq 1$ then C has rational entries. However, z_1, \dots, z_n don't have any branching near b_α , so one should just choose one branch for each z_i .)

Proof: Indeed, the valuation matrix of (z_1, \dots, z_n) is the identity matrix, so

$$\begin{aligned} z_1 &= x_1 h_1; \\ z_2 &= x_2 h_2; \\ &\vdots \\ z_n &= x_n h_n, \end{aligned}$$

where h_1, \dots, h_n are holomorphic non-zero functions in a neighborhood of b_α . Then

$$\frac{\partial z_i}{\partial x_j} \Big|_{b_\alpha} = \delta_{ij} h_i(b_\alpha),$$

so,

$$|J(b_\alpha)| = \left| \left\{ \frac{\partial z_i}{\partial x_j} \right\} \right| = h_1(b_\alpha) \cdots h_n(b_\alpha) \neq 0.$$

Therefore, by the Inverse Function Theory, (z_1, \dots, z_n) are coordinates in a neighborhood of b_α . Also, $\{z_i = 0\} = \{x_i = 0\}$ in a neighborhood of b_α for any $i = 1, \dots, n$. So, (z_1, \dots, z_n) are good coordinates. \square

Let (y_1, \dots, y_n) be generalized local parameters and let $D = \{d_{ij}\}$ be the corresponding valuation matrix. Let \hat{L}^n be the lattice of monomials in y_1, \dots, y_n endowed with the lexicographic order with respect to the basis (y_1, \dots, y_n) (first with respect to y_n ,

then y_{n-1} and so on). Consider the flag $\hat{L}^n \supset \hat{L}^{n-1} \supset \dots \supset \hat{L}^0$ of isolated subgroups in \hat{L}^n . Let $\hat{\mathcal{C}}$ be the corresponding system of cones, \hat{L} be the corresponding semigroup, and $\hat{\mathcal{T}}$ be the corresponding system of toric varieties. Note, that all the toric varieties in $\hat{\mathcal{T}}$ are normal and, moreover, isomorphic to \mathbb{C}^n .

Note that the valuation matrix D and its inverse C are lower-triangular integer matrix with units on the diagonal in this case. Therefore,

$$\begin{aligned} z_1 &= y_1^{c_{11}} \dots y_1^{c_{1n}}; \\ z_2 &= y_1^{c_{21}} \dots y_1^{c_{2n}}; \\ &\vdots \\ z_n &= y_1^{c_{n1}} \dots y_n^{c_{nn}}, \end{aligned}$$

are standard coordinates on a variety $\hat{T}_z \in \hat{\mathcal{T}}$.

Therefore, (z_1, \dots, z_n) provides an isomorphism $\psi^{-1} : W \rightarrow \hat{U}$ between a neighborhood W of $b_\alpha \in \overline{V}_n$ and a neighborhood \hat{U} of the origin in \hat{T}_z . Moreover, $\psi(\hat{U}^k) \subset \overline{V}_k$, where $\hat{U}^k = \hat{T}^k \cap \hat{U}$, and $\psi(\hat{U}^n) = W \setminus \{z_1 \dots z_n = 0\}$.

Let $\hat{\phi} := \pi \circ \psi : \hat{U} \rightarrow V_n$. We proved the following

Theorem 2.4.1. *The map $\hat{\phi}$ has the following properties:*

1. $\hat{\phi}(\hat{U}^k) \subset V_k$ for $k = 0, \dots, n$;
2. $\hat{\phi}|_{\hat{U}^n}$ is an isomorphism to the image;
3. for any hypersurface $H \subset V^n$ one can choose the resolution $\pi : \overline{V}_n \rightarrow V_n$ in such a way, that $H \cap \hat{\phi}(\hat{U}^n) = \emptyset$.

Corollary 1. *Let f be a meromorphic function in a neighborhood of $a \in V^n$. One can choose the variety $\hat{T} \in \hat{\mathcal{T}}$, neighborhood of the origin $\hat{U} \subset \hat{T}$, and the map $\hat{\phi} : \hat{U} \rightarrow V^n$ in such a way that there exist a power series $f_v \in F(\hat{L})$ converging to $f \circ \hat{\phi}$ in \hat{U}^n . In other words, f expands into a power series in (v_1, \dots, v_n) normally converging to f in $\hat{\phi}(\hat{U}^n)$, and the Newton's polyhedron of this power series belong to a cone from $\hat{\mathcal{C}}$, shifted by an integer vector.*

Corollary 2. *Parameters (v_1, \dots, v_n) induce the homomorphism $v^* : F(V^n) \rightarrow F(\hat{L})$ from the field of meromorphic functions on V^n to the field $F(\hat{L})$ of Laurent power series of \hat{L} . In particular, the field of meromorphic functions $F(V^n)$ is endowed with the valuation $\nu_v = \nu_{F(\hat{L})} \circ v^*$.*

Corollary 3. *Let ω be a meromorphic n -form on V_n . Let $\omega = f dv_1 \wedge \dots \wedge v_n$. Then $\text{res}_{(V^n \supset \dots \supset V_0, a_\alpha = \phi_0(b_\alpha))}(\omega) = \text{res}(v^*(f) dv_1 \wedge \dots \wedge dv_n)$.*

Corollary 4. *Let (v'_1, \dots, v'_n) be another set of local parameters. Then the set of Laurent series $(v^*(v'_1), \dots, v^*(v'_n))$ define the change of variables from $\hat{\mathcal{T}}$ to $\hat{\mathcal{T}}'$. Moreover, $v'^* = \psi \circ v^*$, where $\psi : F(\hat{L}) \rightarrow F(\hat{L}')$ is the isomorphism given by the change of variables.*

Corollary 5. *Parshin's residue doesn't depend on the choice of local parameters.*

Note, that $\hat{\phi}|_{\hat{U}^k}$ for $k < n$ is not an isomorphism to the image. It is rather a covering, at least locally (one should be careful defining what this locality actually means). Indeed, $\hat{\phi} = \pi \circ \psi$, where $\psi|_{\hat{U}^k}$ is an isomorphism to the image for all $k = 0, \dots, n$, while $\pi|_{\tilde{V}^k}$ can be a branched covering of a higher degree with some branching near our flag. We want to improve it in such a way that $\phi|_{U^k}$ is an isomorphism to the image for all $k = 0, \dots, n$. In particular, it will help us to understand the local geometry of V^n near the flag $V^n \supset \dots \supset V^0$. In order to do so, one needs to consider special generalized local parameters, which we call *toric parameters* and more general systems of (non-normal) toric varieties, associated to flags of lattices, different from the flag of isolated subgroups.

Let $\rho_k : \tilde{V}_k \rightarrow V_k$ be the normalization maps for $k = 0, \dots, n$ ($\rho_n = p_n$). Let $\hat{V}_{k-1} \subset \tilde{V}_k$ be general primages of hypersurfaces $V_{k-1} \subset V_k$ under ρ_k . For every $k = 1, \dots, n$, let t_k be a meromorphic function on \tilde{V}_k which has zero of order 1 at a general point of \hat{V}_{k-1} . One can think of t_k as a meromorphic function on V_k as well and, moreover, for every k let us continue t_k to a meromorphic function on the whole V_n . For simplicity, we'll denote this continuation by t_k as well.

By applying the Theorem 2.2.2 several times one can get the following diagram:

$$\begin{array}{cccc}
V_n \supset V_{n-1} \supset \dots \supset V_1 \supset V_0 & & & \\
\downarrow \pi_n & \downarrow \pi_n & \downarrow \pi_n & \downarrow \pi_n \\
\vdots & \vdots & \vdots & \vdots \\
\downarrow \pi_3 & \downarrow \pi_3 & \downarrow \pi_3 & \downarrow \pi_3 \\
V_n^2 \supset V_{n-1}^2 \supset \dots \supset V_1^2 \supset V_0^2 & & & \\
\downarrow \pi_2 & \downarrow \pi_2 & \downarrow \pi_2 & \downarrow \pi_2 \\
V_n^1 \supset V_{n-1}^1 \supset \dots \supset V_1^1 \supset V_0^1 & & & \\
\downarrow \pi_1 & \downarrow \pi_1 & \downarrow \pi_1 & \downarrow \pi_1 \\
V_n \supset V_{n-1} \supset \dots \supset V_1 \supset V_0 & & &
\end{array}$$

where $(\pi_1 \circ \pi_2 \circ \dots \circ \pi_k)|_{V_k^k} : V_k^k \rightarrow V_k$ is a resolution of singularities of the flag $V_k \supset \dots \supset V_0$ respecting the functions t_1, \dots, t_k .

Let $b_\alpha \in V_0^n$ as before. Let (x_1, \dots, x_k) be good coordinates in a neighborhood of the $b_\alpha^k := \pi_{k+1} \circ \dots \circ \pi_n(b_\alpha)$. Let $\tilde{t}_i := (\pi_1 \circ \dots \circ \pi_k)^*(t_i)$ for $i = 1, \dots, k$. Let $D^k = \{d_{ij}^k\}$ be the valuation matrix of $(\tilde{t}_1, \dots, \tilde{t}_k)$ with respect to (x_1, \dots, x_k) and let $C^k = \{c_{ij}^k\} = (D^k)^{-1}$ be the inverse matrix.

Lemma 2.4.4. $d_{ij}^k = 0$ for $i < j$, $d_{kk}^k = 1$, and $d_{ii}^k \neq 0$ for $i = 1, \dots, k-1$.

Proof: Similar to Lemma 2.4.2. □

Let

$$\begin{aligned} t_1^k &= t_1^{c_{11}^k} \dots t_1^{c_{1k}^k}, \\ t_2^k &= t_1^{c_{21}^k} \dots t_1^{c_{2k}^k}, \\ &\vdots \\ t_k^k &= t_1^{c_{k1}^k} \dots t_n^{c_{kk}^k}, \end{aligned}$$

and let L^k be the lattice of monomials in (t_1^k, \dots, t_k^k) . Here L_k is a subgroup in the multiplicative group of monomials in (t_1, \dots, t_n) with rational powers. Note, that although t_i^k are multivalued functions globally, they don't any branching near $b_\alpha \in V_n^n$. We choose the branches simultaneously for all $k = 1, \dots, n$. By abuse of notations we denote these branches by t_i^k as well.

Lemma 2.4.5. $L^n \supset \dots \supset L^1 \supset L^0 := \{0\}$ is a flag of lattices, where L^n is endowed with the lexicographic order with respect to the basis (t_1^n, \dots, t_n^n) (first with respect to t_n^n , then t_{n-1}^n , etc.). Moreover, the cone generated by $(t_1^{k-1}, \dots, t_{k-1}^{k-1})$ is a subset of the cone generated by (t_1^k, \dots, t_k^k) .

Proof: It is enough to prove that $(t_1^{k-1}, \dots, t_{k-1}^{k-1})$ are monomials with positive integer powers in $(t_1^k, \dots, t_{k-1}^k)$. Indeed, we have the map $\pi_k|_{V_{k-1}^k} : V_{k-1}^k \rightarrow V_{k-1}^{k-1}$, which is continuous. $(t_1^{k-1}, \dots, t_{k-1}^{k-1})$ are coordinates in a neighborhood of $b_\alpha^{k-1} = \pi_k(b_\alpha^k)$ in V_{k-1}^{k-1} and $(t_1^k, \dots, t_{k-1}^k)$ are coordinates in a neighborhood of b_α^k in V_{k-1}^k . Therefore, the powers have to be positive and integer. \square

Let \mathcal{C} , L , and \mathcal{T} be, respectively, system of cones, semigroup, and system of toric varieties, associated to the flag of lattices $L^n \supset \dots \supset L^1 \supset L^0$.

Note, that t_1^n, \dots, t_n^n are generalized local parameters (here we consider t_1^n, \dots, t_n^n as functions in a neighborhood of $b_\alpha \in V_n^n$). So, one can apply the Theorem 2.4.1 to them.

Note, that lattice $\hat{L}^n = L^n$. So, the flag of isolated subgroups $\hat{L}^n \supset \hat{L}^{n-1} \supset \dots \supset \hat{L}^0$ is the normalization of the flag of lattices $L^n \supset \dots \supset L^1 \supset L^0$. Let $\hat{\phi} : \hat{U} \rightarrow V_n$ be the map from the Theorem 2.4.1. Here $\hat{U} \subset \hat{T}_{t^n} \in \hat{\mathcal{T}}$ is a neighborhood of the origin. Let $T_{t^n} \in \mathcal{T}$ be the variety which normalization is \hat{T}_{t^n} and let $\nu : \hat{U} \rightarrow U$ be the restriction of the normalization map to \hat{U} ($U = \nu(\hat{U})$). Let $U^k = \nu(\hat{U}^k)$ (i.e. the intersection of U with the corresponding toric orbit in T_{t^n}).

Theorem 2.4.2. The map $\hat{\phi} : \hat{U} \rightarrow V_n$ factors through the normalization map $\nu : \hat{U} \rightarrow U$, i.e. $\hat{\phi} = \phi \circ \nu$ where $\phi : U \rightarrow V_n$. Moreover, $\phi|_{U^k}$ is an isomorphism to the image inside V_k for all $k = 0, 1, \dots, n$.

Proof: According to the Lemma 1.3.5 it is enough to show that $\hat{\phi}$ factors through the normalization on the level of sets and that ϕ is continuous (then it is regular). More over, similarly to the Theorem 1.4.1 the continuity follows immediately. The only thing we need to check is that ϕ is well defined on the level of sets, i.e. that for any $x \in U$ and $y_1, y_2 \in \nu^{-1}(x)$ we have $\hat{\phi}(y_1) = \hat{\phi}(y_2)$. Indeed, let k be the

smallest number such that $x \in \overline{U^k}$. The functions t_k^1, \dots, t_k^k provides coordinates on V_k^k near b_α^k . On the other hand, they belong to L_k and, therefore, they are regular on $U \setminus \overline{U^{k-1}}$, in particular, at x .

Note, that $\hat{\phi} = \pi_1 \circ \dots \circ \pi_n \circ \psi$. Denote $z_i = \pi_{k+1} \circ \dots \circ \pi_n \circ \psi(y_i) \in V_k^k$. It follows from the above, that t_k^1, \dots, t_k^k don't distinguish z_1 and z_2 . Therefore, $z_1 = z_2$.

Consider now the restriction $\phi|_{U^k}$. It follows from the above that $\phi|_{U^k} = \pi_1 \circ \dots \circ \pi_k \circ \psi_k$, where ψ_k is a regular map from U^k to the complement of the coordinate cross in a neighborhood of b_α^k in V_k^k . Moreover, ψ_k is an isomorphism to the image (since $dt_k^1 \wedge \dots \wedge dt_k^k$ is a non-degenerate top form both in the image and the preimage). $\pi_1 \circ \dots \circ \pi_k$ is an isomorphism to the image on the complement to the exceptional divisor. Therefore, $\phi|_{U^k}$ is an isomorphism to the image as well. \square

References

- [BiM] Bierstone, Edward; Milman, Pierre D. Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. *Invent. Math.* 128 (1997), no. 2, 207-302.
- [BrM] Brylinski, J.-L.; McLaughlin, D. A. Multidimensional reciprocity laws. *J. Reine Angew. Math.* 481 (1996), 125-147.
- [H] Hironaka, Heisuke. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2)* 79 (1964), 109-326.
- [L] Lomadze, V. G. On residues in algebraic geometry. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* 45 (1981), no. 6, 1258-1287.
- bibitem[P1]P1 Parshin, A. N. On the arithmetic of two-dimensional schemes. *Math. SSSR Izv.* 10 (1976), no. 4, 695-729.
- [P2] Parshin, A. N. Chern classes, adèles and L -functions. *J. Reine Angew. Math.* 341 (1983), 174-192.