PROBLEMS

MATT KOSTER

Definition. Let $p(x)$ be a polynomial with coefficients in a field $\mathbb{F}$. We say that $p(x)$ *splits* if it can be written:

$$p(x) = (x - a_1) \cdots (x - a_n)$$

where $a_i \in \mathbb{F}$.

Problems 1.

(1) Prove that $p(x) = x^2 + 1$ does not split over $\mathbb{R}$.
(2) Prove that $x^2 + x + 1$ does not split over $\mathbb{F}_2$.
(3) Prove that for every $T : V \to V$ such that $p_T(x)$ splits, T is similar to an upper triangular matrix.
(4) Deduce that every matrix $A \in M_n(\mathbb{C})$ is similar to an upper triangular matrix.
(5) If A is an upper triangular matrix and the diagonal entries are $a_1, a_2, ..., a_n$ prove that the diagonal entries of A^k are $a_1^k, a_2^k, ..., a_n^k$.

Definition Let $A \in M_n(\mathbb{F})$ be an $n \times n$ matrix. We define the *matrix exponential* of A, or simply the *exponential* of A, denoted $\exp(A)$ or e^A by:

$$\exp(A) = e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

Problems 2.

(1) Suppose U is an upper triangular matrix with diagonal entries $a_1, ..., a_n$. Show that the diagonal entries of U are equal to $e^{a_1}, e^{a_2}, ..., e^{a_n}$.
(2) If Q is invertible prove that $\exp(QAQ^{-1}) = Qe^AQ^{-1}$ for any $A \in M_n(\mathbb{F})$.
(3) If $A \in M_n(\mathbb{C})$ has $\text{trace}(A) = 0$, use problems 1 and 2 together with problem 4 from the first set of problems to prove that $\det(e^A) = 1$.

Remark. Actually more is true. *Jacobi's identity* says that for any A, $\det(e^A) = e^{\text{Tr}(A)}$ from which the above result follows and shows moreover that e^A is invertible for any A.

Definition. A *Lie algebra* over $\mathbb{F}$ (or an $\mathbb{F}$-*Lie algebra*) is a vector space $\mathfrak{g}$ over $\mathbb{F}$ together with bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that:

(1) $[X, X] = 0$ for all $X \in \mathfrak{g}$ (ie. $[\cdot,\cdot]$ is alternating).
(2) $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$ for all $X, Y, Z \in \mathfrak{g}$ (called the *Jacobi identitiy*).

An alternating bilinear map is automatically skew-symmetric:

$$[X, Y] = -[Y, X]$$

and a skew symmetric bilinear map is alternating if $\mathbb{F}$ has characteristic not equal to 2 (see tutorial 8).

Problems 3.

(1) Prove that $\mathfrak{gl}(n; \mathbb{R}) := M_n(\mathbb{R})$ together with the bracket $[X, Y] = XY - YX$ is a real Lie algebra. What is the dimension of $\mathfrak{gl}(n; \mathbb{R})$?
(2) Prove that $\mathfrak{sl}(n; \mathbb{R}) := \{X \in M_n(\mathbb{R}) : \text{trace}(X) = 0\}$ together with the bracket $[X, Y] = XY - YX$ is a real Lie algebra (called the *special linear Lie algebra*). What is the dimension of $\mathfrak{sl}(n; \mathbb{R})$?
(3) Let V be any vector space of dimension n. Prove that V together with the bracket $[X, Y] = 0$ is a Lie algebra. This is called the *abelian Lie algebra of dimension n.*
(4) Prove that $\mathfrak{so}(n, \mathbb{R}) := \{X \in M_n(\mathbb{R}) : X^t = -X\}$ together with the bracket $[X, Y] - XY - YX$ is a real Lie algebra (called the *special orthogonal Lie algebra*). What is the dimension of $\mathfrak{so}(n; \mathbb{R})$?
(5) Prove that the *Heisenberg algebra* $\mathfrak{h}(3; \mathbb{R})$ defined by:

$$\mathfrak{h}(3; \mathbb{R}) := \{\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}\}$$

together with the bracket $[X, Y] = XY - YX$ is a real Lie algebra. What is the dimension of $\mathfrak{h}(3; \mathbb{R})$?
(6) Let $\mathfrak{h} := \mathfrak{h}(3; \mathbb{R})$ as in the previous problem.and let:

$$L := \{\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ z & y & 0 \end{pmatrix}\}$$

Prove that the map $f : L \to \mathfrak{h}^*$ given by $f(X)(Y) = \text{Tr}(XY)$ is a natural isomorphism.
(7) Prove that $V = \mathbb{R}^3$ together with the bracket $[X, Y] = X \times Y$ (the *cross product* of vectors in $\mathbb{R}^3$) is a Lie algebra.
(8) Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras with brackets $[\cdot,\cdot]_{\mathfrak{g}}, [\cdot,\cdot]_{\mathfrak{h}}$. A linear transformation $T : \mathfrak{g} \to \mathfrak{h}$ that also satisfies $T([X, Y]_{\mathfrak{g}}) = [T(X), T(Y)]_{\mathfrak{h}}$ is called a *Lie algebra homomorphism.* If T is a vector space isomorphism that is also a Lie algebra homomorphism we say that T is an *isomorphism of Lie algebras* or a *Lie algebra isomorphism* and if such a T exists we say $\mathfrak{g}$ for all $X, Y \in \mathfrak{g}$ is *isomorphic to* $\mathfrak{h}$ *as Lie algebras* and write $\mathfrak{g} \cong \mathfrak{h}$. Prove that the Lie algebra from the previous example is isomorphic to $\mathfrak{so}(3, \mathbb{R})$.
(9) Let V a finite dimensional vector space over $\mathbb{R}$. Prove that $\bigwedge(V)$ together with $[X, Y] = X \wedge Y - Y \wedge X$ is a real Lie algebra.
(10) Let $SL(2; \mathbb{C}) \subset M_2(\mathbb{C})$ denote the 2×2 matrices A with $\det(A) = 1$. Prove that $\exp : \mathfrak{sl}(2; \mathbb{C}) \to SL(2; \mathbb{C})$ is surjective (how do we know that exp maps into this space at all?) (hint: Jordan canonical form and the first two blocks of exercises).

(11) If $A \in \mathfrak{sl}(2;\mathbb{R})$ and the characteristic polynomial of A does not split, it is a fact that A is similar is similar to a matrix of the form $\begin{pmatrix} 0 & \det(A) \\ -1 & 0 \end{pmatrix}$ (this is the *rational canonical form*). Use this together with the Jordan canonical form to prove that $\det(e^A) = 1$. so exp maps $\mathfrak{sl}(2;\mathbb{R})$ into $SL(2;R)$ similar to the previous exercise.

(12) Prove that $\exp : \mathfrak{sl}(2;\mathbb{R}) \to SL(2;\mathbb{R})$ is **not** surjective. (Hint: prove that $A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ is not in the image. What is the characteristic polynomial of this matrix? Does it split? If $A = \exp(B)$ what are the eigenvalues of B?)

Definitions.

(1) For $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{g}$ define:

$$[\mathfrak{a}, \mathfrak{b}] = \{\sum_{i=1}^{n} [A_i, B_i] \ : \ A_i \in \mathfrak{a}, B_i \in \mathfrak{b}, \ n \in \mathbb{N}\} = \text{span}\{[A, B], A \in \mathfrak{a}, b \in \mathfrak{b}\}$$

(2) We say that $\mathfrak{h} \subseteq \mathfrak{g}$ is a *Lie subalgebra* if $\mathfrak{h}$ is a vector subspace and moreover $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$.

(3) A subspace $\mathfrak{i} \subseteq \mathfrak{g}$ is called an *ideal* if $\mathfrak{i}$ is a Lie subalgebra and $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}$ and write $\mathfrak{i} \trianglelefteq \mathfrak{g}$.

Problems 4.

(1) Prove that $[\mathfrak{g}, \mathfrak{g}] \trianglelefteq \mathfrak{g}$.
(2) Prove that $\{0\} \trianglelefteq \mathfrak{g}$.
(3) If $\mathfrak{g}$ is the abelian Lie algebra of dimension n find all of the ideals of $\mathfrak{g}$.
(4) Prove that the only 1-dimension Lie algebra is the abelian one.
(5) Find $[\mathfrak{g}, \mathfrak{g}]$ for the following (hint: picking bases is useful):
  (a) $\mathfrak{g} = \mathfrak{h}(3, \mathbb{R})$
  (b) $\mathfrak{g} = \mathfrak{sl}(2; \mathbb{R})$
  (c) $\mathfrak{g} = \mathfrak{sl}(2; \mathbb{C})$
  (d) $\mathfrak{g} = \mathfrak{so}(3; \mathbb{R})$
  (e) $\mathfrak{g} = [\mathfrak{h}, \mathfrak{h}]$ where $\mathfrak{h}$ is the Lie algebra computed in part (a).

Definitions

(1) The *derived series of* $\mathfrak{g}$ is the following sequence $(\mathfrak{g}^{(n)})$ of subalgebras:

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}], \quad \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}], ...$$

and we say that $\mathfrak{g}$ is *solvable* if $\mathfrak{g}^{(k)} = 0$ for some k.

(2) The *lower central series of* $\mathfrak{g}$ is the following sequence $(\mathfrak{g}_{(n)})$ of subalgebras:

$$\mathfrak{g}_{(0)} = \mathfrak{g}, \quad \mathfrak{g}_{(1)} = [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}_{(0)}, \mathfrak{g}], \quad \mathfrak{g}_{(2)} = [\mathfrak{g}_{(1)}, \mathfrak{g}], ...$$

and we say that $\mathfrak{g}$ is *nilpotent* if $\mathfrak{g}_{(k)} = 0$ for some k.

Problems 5.

(1) Prove that every nilpotent Lie algebra is solvable.
(2) Prove that every abelian Lie algebra is nilpotent (hence solvable).
(3) Prove that every 2-dimension Lie algebra is solvable.
(4) Prove that there exists two non isomorphic 2-dimension Lie algebras.
(5) Prove that every 2-dimension Lie algebra is isomorphic to one of the two from the previous problem.
(6) Determine which of the Lie algebras from problems 4 question 5 are solvable.

Definitions.

(1) A Lie algebra is called *simple* if it is not abelian and has no nonzero ideals.

(2) Given two Lie algebras $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}), (\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$ we define their *direct sum* $\mathfrak{g} \oplus \mathfrak{h}$ to be the Lie algebra whose underlying vector space is the vector space direct sum, together with the bracket $[\cdot,\cdot]$ given by:

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2]_{\mathfrak{g}}, [Y_1, Y_2]_{\mathfrak{h}})$$

(3) A Lie algebra is called *semisimple* if it is the direct sum of simple Lie algebras.

Problems.

(1) Prove that every simple Lie algebra is semisimple.
(2) Show that $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g}$ has no nonzero abelian ideals.
(3) If $\mathfrak{g}$ is semisimple show that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.
(4) Determine which of the Lie algebras from problems 4 question 5 are semisimple.