WHAT IS a Tropical Curve?

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A tropical curve is an algebraic curve defined over the semifield $\mathbb{T}$ *of tropical numbers.* The goal of this note is to make sense out of this phrase.

1. TROPICAL BACKGROUND

We define $\mathbb{T}$ to be the set $\mathbb{R} \cup \{-\infty\}$ and we equip it with the *"addition"* operation "$x + y$" $= \max\{x, y\}$ and with the "multiplication" operation "xy" $= x + y$. We use the quotation marks to distinguish between the standard and tropical arithmetic operations. Our "additive zero" is $-\infty$ while "multiplicative unit" is 0.

The term *tropical* is taken from Computer Science, where it was coined to commemorate contribution of the Brazilian school. The term *semifield* refers to the properties of the tropical arithmetic operations: we have all the axioms required for a *field*, except for the absence of subtraction (as our addition is idempotent "$x + x$" $= x$). Luckily, one does not need subtraction to write down polynomials (they are *sums* of monomials)!

2. CURVES IN $\mathbb{R}^2$

Consider a polynomial in two variables

$$p(x, y) = \text{``}\sum_{j,k} a_{jk} x^j y^k\text{''} = \max_{j,k}(jx + ky + a_{jk}).$$

The tropical curve C defined by p consists of those points $(x, y) \in \mathbb{R}^2$ where p is not differentiable. In other words, C is the locus where the maximum is assumed by more than one "monomials" of p. It is easy to see that $C \subset \mathbb{R}^2$ is a graph and its edges are straight intervals with rational slopes, see Figure 1.

We have "$a_{j_1k_1} x^{j_1} y^{k_1}$" $=$ "$a_{j_2k_2} x^{j_2} y^{k_2}$" $>$ "$a_{jk} x^j y^k$", at the edge $E \subset C$. Thus E is perpendicular to the vector $(j_1 - j_2, k_1 - k_2)$. We can enhance the edge E with a natural number $w(E)$ (called its *weight*) equal to $\mathrm{GCD}(j_1 - j_2, k_1 - k_2)$.

Take a vertex $A \in C$ and consider the edges $E_1, \ldots, E_n$ adjacent to A. Let $v(E_j) \in \mathbb{Z}^2$ be the primitive integer vector from A in the direction of E_j. It is easy to see that we have the following *balancing* (or zero-tension) condition at each vertex of C:

$$\sum_{j=1}^{n} w(E_j) v(E_j) = 0. \tag{1}$$

Furthermore, one can easily show that any weighted piecewise-linear graph in $\mathbb{R}^2$ with rational slopes of the edges and with the zero-tension condition at the vertices is given by some tropical polynomial.

3. PROJECTIVE SPACE $\mathbb{TP}^n$

The plane $\mathbb{R}^2$ can be thought of as a part of the tropical affine plane $\mathbb{T}^2 = [-\infty, +\infty)^n$. The regular function on $\mathbb{T}^2$ are tropical polynomials, the regular functions on $\mathbb{R}^2 = (\mathbb{T}^\times)^2$ are tropical Laurent polynomial. Note that a monomial is an affine-linear function with an integer slope and therefore the geometric structure on $\mathbb{R}^2$ encoding the tropical structure is the $\mathbb{Z}$-*affine structure.*

In its turn, the plane $\mathbb{T}^2$ can be compactified to the projective plane $\mathbb{TP}^2$. To construct $\mathbb{TP}^2$ we take the quotient of $\mathbb{T}^3 \smallsetminus \{(-\infty, -\infty, -\infty)\}$ by the usual equivalence relation $(x, y, z) \sim (\lambda x, \lambda y, \lambda z)$, $\lambda \neq 0$. As in the classical case we have three affine charts, so $\mathbb{TP}^2$ can be obtained by gluing three copies of $\mathbb{T}^2$. Thus we may think of $\mathbb{TP}^2$ as a triangle-like compactification of $\mathbb{R}^2$ taken with the tautological $\mathbb{Z}$-affine structure. Each side of the triangle corresponds to a copy of $\mathbb{TP}^1$ (which is itself a compactification of $\mathbb{R}$ by two points). Similarly, we may define $\mathbb{TP}^n \supset \mathbb{R}^n$ as well as other tropical toric varieties.

4. PROJECTIVE CURVES

We have compact tropical curves in $\mathbb{TP}^n \supset \mathbb{R}^n$. Let Γ be a finite graph and $h : \Gamma \to \mathbb{TP}^n$ be a continuous map that takes the interior of every edge E to a straight (possibly unbounded) interval with a rational slope in $\mathbb{R}^n$. If we can prescribe a positive integer weight to each edge so that (1) holds at every vertex in $\mathbb{R}^n$ then we say that $h : \Gamma \to \mathbb{TP}^n$ is a tropical curve.

The *degree* d of $h(\Gamma)$ is the intersection number with any of the $(n + 1)$ $\mathbb{TP}^{n-1}$-divisors at infinity. This can be done by examining the unbounded edges $E_1, \dots , E_l$. E.g. the intersections with the "last" infinity divisor $D_\infty = \mathbb{T}P^n \smallsetminus \mathbb{T}^n$ is given by those E_j whose outward primitive vectors $v(E_j)$ have at least one of its coordinate positive; the local intersection number with D_∞ is $w(E_j)$ times that coordinate (assuming that it is maximal) and d is the sum of these local intersection numbers. The balancing condition ensures that the total intersection number with all infinity divisors is the same.

The *genus* of Γ is $g = \dim H_1(\Gamma)$. Figure 1 depicts a line (i.e. a degree 1 curve) and a conic (i.e. a degree 2 curve) in $\mathbb{TP}^2$. Both of these curves are rational (i.e. their genus is zero).

5. Enumerative geometry

Note that tropical curves behave quite similar to classical algebraic curves. Prove (as an exercise) that any two points in $\mathbb{TP}^n$ can be connected with a line. Curves in $\mathbb{TP}^n$ of degree d and genus g vary in a family of dimension at least $(n+1)d+(n-3)(1-g)$. In many cases this lower bound is exact: for instance if $g = 0$ (for any n) or if $n = 2$ (for any g) if h is an immersion.

E.g., if we fix a configuration $\mathcal{C}$ of $3d - 1 + g$ generic points in $\mathbb{TP}^2$ then only finitely many curves $h_j : \Gamma_j \to \mathbb{TP}^2$ of degree d and genus g will pass through $\mathcal{C}$. Unlike the complex coefficients case the actual number of such tropical curves h_j will depend on the choice of $\mathcal{C}$. However, each such tropical curve comes with a combinatorial multiplicity so that the number of curves with multiplicity is invariant. And this invariant coincides with the number of complex curves of degree d and genus g passing through a generic configuration of $3d - 1 + g$ points in $\mathbb{CP}^2$ and gives an efficient way of computing that number.

There is also a different choice of multiplicities for h_j that is responsible for enumeration over $\mathbb{R}$ (some real curves are counted with the sign $+1$ and some with -1 and their total number is different from the complex counterpart). Sum of the real multiplicities of $h_j(\Gamma_j)$ gives the answer for the corresponding real enumerative problem.

As an example consider the case $d = 3$ and $g = 0$ (see Figure 2). We fix a configuration $\mathcal{C}$ of 8 points in $\mathbb{R}^2 \subset \mathbb{TP}^2$. Depending on the choice of $\mathcal{C}$ there might be 9 or 10 tropical curves via $\mathcal{C}$. However, the sum of their complex multiplicities is always 12 while the sum of their real multiplicities is always 8.

6. Tropical curves as metric graphs

A map $h : \Gamma \to \mathbb{TP}^n$ can be used to induce tropical structure on Γ. Tropical monomials p in $\mathbb{R}^n = (\mathbb{T}^\times)^n$ give smooth functions on every edge $E \subset \Gamma$ and can be used to measure the length of E. Define the length of E to be the smallest of such lengths divided by the weight of E. This turns Γ into a metric graph: the leaves (i.e. the edges adjacent to 1-valent vertices) will have infinite length while the inner edges will have finite length.

Conversely, a metric graph structure on Γ can be used to define *tropical* maps $\Gamma \to X$ where X is $\mathbb{T}^n$, $\mathbb{TP}^n$ or any other tropical variety (which can be defined in higher dimension as a polyhedral complex equipped with an integer affine structure, it was the specific of dimension 1 that allowed us to reformulate the integer affine structure). Higher weight appears when h "stretches" the edges by an integer amount.

There is an equivalence relation between tropical curves generated by the following relation: at any point $x \in \Gamma$ we may introduce an infinite length interval connecting x with a new 1-valent vertex. This equivalence allows to turn a map given by regular functions to a tropical morphism. Also it allows to treat any marked point as a 1-valent vertex. This turns, e.g. the space $\mathcal{M}_{0,n}$ of trees with n marked points into an $(n - 3)$-dimensional tropical variety.

Most classical theorems on Riemann surfaces have counterparts for tropical curves, in particular, the Abel-Jacobi theorem, the Riemann-Roch theorem and the Riemann theorem on θ-function. Furthermore, the tropical curves can be obtained as degeneration of the complex structure on Riemann surfaces. This degeneration provides a link between classical and tropical geometry useful for applications. Many features of complex and real curves become easily visible after tropicalization.

7. Further reading

[1] Mikhalkin, G., Enumerative tropical algebraic geometry in $\mathbb{R}^2$. *J. Amer. Math. Soc.* 18 (2005), no. 2, 313–377.

[2] Mikhalkin, G., Zharkov, I., Tropical curves, their Jacobians and Theta-functions, 2006.

[3] Richter-Gebert, J., Sturmfels, B., Theobald, Th., First steps in tropical geometry. Contemp. Math., 377, 2005, 289–317.