Tropical geometry

Grigory Mikhalkin

Contents

CHAPTER 1

Introduction

1. Overview

Algebraic Geometry provides a uniform approach to some topologically very distinct situations. As an example, let us consider *a line in the affine 2-plane*. Topologically this set-up only makes sense if we fix the ground field, i.e. the possible values for the coordinates in the 2-plane. If the ground field is $\mathbb{R}$ we have the "most classical" situation: the plane is indeed a real plane $\mathbb{R}^2$ and the line is a real line $\mathbb{R}$.

For the other choices of ground fields the topological picture is different, e.g. the complex plane $\mathbb{C}^2$ is a 4-manifold while over finite fields we do not have any interesting topology at all. In the same time despite such differences the behavior of lines remain the same. Namely, via any pair of distinct points in the plane we can draw a unique line. Also, any pair of lines intersect in a single point (unless they are parallel). This behavior is dictated by the algebra of linear equations.

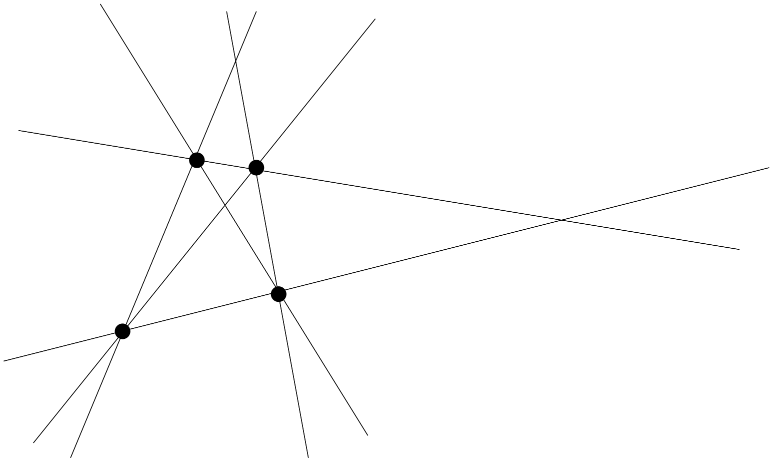

FIGURE 1. The three new intersection points are collinear according to the Fano axiom

Some other properties of lines in the plane depend on the choice of the ground field. A famous example is the *Fano axiom*. Given any quadruple of distinct points in the plane we may consider the triple of points obtained as the intersections of the pairs of lines corresponding to all possible choices of two disjoint pairs among the four initial points. The Fano axiom states that the resulting three points are collinear. This axiom clearly does not hold for $\mathbb{C}$ or $\mathbb{R}$, but it holds for the fields of characteristic 2.

When we pass to tropical geometry the ground field gets replaced with the tropical semifield $\mathbb{T}$ (which we introduce in the next section) with limited arithmetics and algebra. E.g., it becomes no longer clear how to even define the *characteristic* of $\mathbb{T}$. Meanwhile, such geometric objects as points, lines, etc. are perfectly well-defined. In particular, the Fano axiom still holds in tropical geometry. More general, in tropical geometry we may find reflections of properties from rather different fields with different algebraic origins.

Most algebraic constructions are obstructed by the absence of subtraction in $\mathbb{T}$. In the same time, geometry not only remains equally transparent, but it gets more explicit and visual. The goal of this book is to justify this statement.

2. The tropical semifield $\mathbb{T}$

Definition 1.1. A commutative *semiring* is a set equipped with commutative and associative operations of addition and multiplication so that the distribution law holds while the addition and multiplication operations both have neutral elements. A commutative semiring R is called a *semifield* if the non-zero elements of R form a group (denoted with $R^\times$) with respect to multiplication.

Example 1.2. The non-negative numbers $\mathbb{R}_{\geq 0}$ equipped with the usual addition and multiplication form a semifield. Its multiplicative group is the group of positive numbers $\mathbb{R}_{>0}$.

The semifield introduced in the following definition is crucial for this book.

Definition 1.3. The tropical semifield $\mathbb{T}$ is the set $\mathbb{R} \cup \{-\infty\}$ equipped with the following two arithmetic operations called *tropical addition* and *tropical multiplication.* If $a, b \in \mathbb{T}$ we set

$$\text{“}a + b\text{”} = \max\{a, b\}$$

and

$$\text{“}ab\text{”} = a + b.$$

The quotation marks are used to signify that the arithmetic operations we are referring to are tropical. It is easy to check that the usual commutativity, associativity and the distribution law hold in tropical arithmetics. Namely, we have “$a + b$” = “$b + a$”, “$(a + b) + c$” = “$a + (b + c)$”, “ab” = “ba”, “$(ab)c$” = “$a(bc)$” and “$a(b + c)$” = “$ab + ac$” for any $a, b, c \in \mathbb{T}$. The element $-\infty = 0_{\mathbb{T}}$ is the additive zero while $0 = 1_{\mathbb{T}}$ is the multiplicative unit, "$0_\mathbb{T} + a$" $= \max\{-\infty, a\} = a$, "$1_\mathbb{T} b$" $= 0 + b = b$, for any $a \in \mathbb{T}$, $b \in \mathbb{T}^\times = \mathbb{T} \smallsetminus \{-\infty\}$. In addition we have "$-\infty a$" $= -\infty$ for any $a \in \mathbb{T}$. However, in contrast with the classical addition the tropical addition is idempotent:

$$\text{“}x + x\text{”} = x.$$

This property makes tropical subtraction impossible, $\mathbb{T}$ is only a semigroup with respect to addition. On the other hand, the non-zero elements $\mathbb{T}^\times = \mathbb{T} \smallsetminus \{-\infty\}$ form a group (isomorphic to $\mathbb{R}$) with respect to multiplication and we have tropical division

$$\text{“}a/b\text{”} = a - b$$

as long as $b \neq -\infty$.

Note that the semifield $\mathbb{T}$ has a natural (Euclidean) topology coming from the identification of $\mathbb{T}$ with the half-open infinite interval $[-\infty, +\infty)$. This topology is natural from the algebraic point of view. Indeed, the Euclidean topology on $[-\infty, +\infty)$ is generated by the sets $\{x \in \mathbb{T} \mid x > a\}$ and $\{x \in \mathbb{T} \mid x < b\}$ for $a, b \in \mathbb{T}^\times = (-\infty, +\infty)$.

Each inequality can be rephrased in agrebraic terms. Indeed the inequality $a \leq b$ for $a, b \in \mathbb{T}$ is equivalent to the identity "a+b=b".

3. The affine space $\mathbb{T}^n$ and the torus $(\mathbb{T}^\times)^n \approx \mathbb{R}^n$

We define the tropical affine n-space as a topological space by

$$\mathbb{T}^n = [-\infty, +\infty)^n.$$

Accordingly, we define the n-torus there

$$(\mathbb{T}^\times)^n = (-\infty, +\infty) = \mathbb{R}^n \subset \mathbb{T}^n.$$

This definition immediately gives the topology on $\mathbb{T}^n$. The algebrogeometric structure is given by *regular functions* on $\mathbb{T}^n$ which come from *tropical polynomials.*

DEFINITION 1.4. A tropical polynomial $f : \mathbb{T}^n \to \mathbb{T}$ is a function given by

$$f(x_1, \dots, x_n) = \text{“}\sum_{j_1,\dots,j_n} a_{j_1\dots j_n} x_1^{j_1} \dots x_n^{j_n}\text{”},$$

where $a_{j_1\dots j_n} \in \mathbb{T}$, the indices j_k are positive integers and the sum is finite.

Let us find the geometric structure on $\mathbb{T}^n$ that would enable us to distinguish tropical polynomials from other continuous functions without a reference to arithmetic operations in $\mathbb{T}$. For that we restrict our attention to the torus $(\mathbb{T}^\times)^n$.

Note that if $x \in \mathbb{T}^\times$ then negative powers "x^{-k}" = "$\frac{1}{x^k}$" $= -kx$ also make sense. Thus we also have the *Laurent polynomials* $(\mathbb{T}^\times)^n \to \mathbb{T}$ defined by

$$\text{“} \sum_{j_1,\dots,j_n} a_{j_1\dots j_n} x_1^{j_1} \dots x_n^{j_n} \text{”},$$

where $a_{j_1\dots j_n} \in \mathbb{T}$, $j_k \in \mathbb{Z}$ and the sum is still finite.

Each monomial

$$\text{“}a_{j_1\dots j_n} x_1^{j_1} \dots x_n^{j_n}\text{”} = j_1 x_1 + \dots + j_n x^n$$

is an affine-linear function in $(\mathbb{T}^\times)^n = \mathbb{R}^n$. Furthermore, the *slope* of this function is $(j_1, \dots, j_n)$ and thus it is integer. The geometric structure that underlies such affine-linear functions is the *integer affine structure.*

4. Integer affine structures on smooth manifolds

Definition 1.5. Let M be a smooth n-dimensional manifold. An integer affine structure on M consists of an open covering U_α and charts $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ such that for each α, β the overlapping map $\phi_\beta \circ \phi_\alpha^{-1}$ can be obtained as the restrictions of an integer affine-linear transformation $\Phi_{\beta\alpha} : \mathbb{R}^n \to \mathbb{R}^n$. Here a map $f : \mathbb{R}^n \to \mathbb{R}^m$ is called integer affine-linear if it is a composition of a $\mathbb{Z}$-linear map $\mathbb{R}^n \to \mathbb{R}^m$ (i.e. a map given by $m \times n$ matrix with integer values) and a translation by an arbitrary vector in $\mathbb{R}^m$. The map f is called an integer affine-linear transformation of $\mathbb{R}^n$ if it is invertible in the class of integer affine-linear maps (note that the invertibility implies that $m = n$).

The manifold M equipped with such structure is called an integer affine manifold. As with all geometric structures of such kind we have *the developing map.* Namely, if $x \in U_\alpha \subset M, y \in U_\beta \subset M$ and $\gamma : [0,1] \to M$ is a continuous path connecting x and y then we have the map $\Phi^\gamma_{\alpha\beta} : \mathbb{R}^n \to \mathbb{R}^n$ defined as follows.

The path $\gamma([0,1])$ can be covered by a finite number of the charts U_{α_j}, $j = 0, \dots k$. We can make sure that $U_{\alpha_{j-1}} \cap U_{\alpha_j} \cap \gamma([0,1]) \neq \emptyset$ for $j > 0$ so that $\alpha_0 = \alpha$ and $\alpha_k = \beta$. Then we define

$$\Phi^\gamma_{\beta\alpha} = \Phi_{\alpha_k\alpha_{k-1}} \circ \dots \circ \Phi_{\alpha_1\alpha_0}.$$

It is easy to see that $\Phi^\gamma_{\beta\alpha}$ depends only on α, β and γ but not on the choice of U_{α_j}. Furthermore, $\Phi^\gamma_{\beta\alpha}$ depends only on the relative homotopy class of the path γ.

Recall that if we fix $x \in M$ then a point in the total space $\tilde{M}$ of the universal covering $\pi : \tilde{M} \to M$ corresponds to a pair $(y, [\gamma])$, where $[\gamma]$ is the relative homotopy class of a path from x to y. Thus if we fix x and α then we get a well-defined map $\delta : \tilde{M} \to \mathbb{R}^n$ by setting

$$\delta(y, [\gamma]) = \Phi^\gamma_{\beta\alpha} \circ \Phi_\alpha,$$

where U_β is chart containing y.

Definition 1.6. The map δ is called the developing map.

As it is easy to see the value $\delta(y, [\gamma]) \in \mathbb{R}^n$ does not depend on the ambiguity in the choice of β. By construction, the developing map is always an open embedding.

Definition 1.7. The integer affine structure on a smooth manifold is called *complete* if the developing map is proper.

Clearly, the product $M \times N$ of two integer affine manifolds M and N has a natural integer affine structure.

Proposition 1.8. *The product $M \times N$ is complete if and only if both M and N are complete.*

Proof. The proposition easily follows from the observation that the universal covering of $M \times N$ can be obtained by taking the product of the universal coverings for M and N. $\square$

For $\mathbb{R}^n$ we have a notion of affine-linear functions with integer slopes or simply integer affine-linear functions. These are the functions obtained from linear maps $\mathbb{R}^n \to \mathbb{R}$ defined over $\mathbb{Z}$ (i.e. such that the image of the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ is integer) after adding an arbitrary constant. Clearly, the pull-back of an integer affine-linear function under an integer affine-linear map $\mathbb{R}^n \to \mathbb{R}^n$ is another integer affine-linear function on $\mathbb{R}^n$.

Furthermore, for any open subset U of an integer affine manifold M we have a well-defined notion of an integer affine-linear function $f : U \to \mathbb{R}$. By definition it is a function that corresponds to an affine-linear function with integer slope on $\mathbb{R}^n$ in each chart. These functions correspond to tropical monomials. While the choice of presentation as a tropical monomial depends on the choice of chart, these functions always correspond to *some* tropical monomials in any chart. Taking the maximal value of integer affine-linear function produces tropical (Laurent) polynomials. Thus geometrically, the tropical structure on $(\mathbb{T}^\times)^n$ may be rephrased as an integer affine-linear structure on $\mathbb{R}^n$.

Recall that the differential of the integer affine-linear transformations in $\mathbb{R}^n$ is defined over $\mathbb{Z}$. Thus an integer tangent vector is mapped to an integer tangent vector. Thus for any integer affine (smooth) manifold M and any point $x \in M$ we have a well-defined integer lattice in the tangent space $T_x M$. This lattice varies smoothly from point to point.

Conversely, if we have a smooth manifold with a coherent choice of integer lattice in the tangent bundle then it does not necessarily come locally from the tautological integer affine structure on $\mathbb{R}^n$ as this is a subject to certain *integrality condition*. Locally such choice of lattice corresponds to finding n linearly independent vector fields on $\mathbb{R}^n$. The integrality condition is the (pairwise) commuting of these vector fields.

Remark 1.9. We see that integer affine-linear smooth manifolds locally can be considered as examples of tropical varieties (as they locally coincide with $(\mathbb{T}^\times)^n$. Similarly, smooth manifolds with a coherent (but not necessarily integrable) choice of integer lattice in the tangent bundle can be considered as examples of *almost tropical* varieties.

5. Morphisms and isomorphisms of integer affine manifolds

Let M and N be integer affine varieties of dimensions m and n. A map $f : M \to N$ is called an integer affine-linear map (or just *morphism* of integer affine-linear varieties) if it is smooth and its differential maps any integer vector tangent to M at any point x to an integer vector (tangent to N at $f(x)$).

Consider a morphism $f : M \to N$ of integer affine-linear varieties, a point $x \in M$ and any pair of charts $U_\alpha \ni x$, $V_\beta \ni f(x)$.

Proposition 1.10. *The map $\psi_\beta^{-1} \circ f \circ \phi_\alpha$ is the restriction to the domain where it is defined (i.e. to $U_\alpha \cap f^{-1}(V_\beta)$) of an integer affine linear map $\mathbb{R}^m \to \mathbb{R}^n$.*

Proof. It suffices to show that if $f : \mathbb{R}^m \to \mathbb{R}^n$ is a map whose differential takes integer vectors to integer vectors then f is integer affine linear. Applying a translation if needed we may assume that f takes the origin of $\mathbb{R}^m$ to the origin of $\mathbb{R}^n$.

We claim that the differential $(df)_0$ of such f at the origin coincides with the map itself (after the natural identification of $\mathbb{R}^m$ with the tangent space at its origin). The integrality assumption assures that $(df)_0$ is defined over $\mathbb{Z}$. By the continuity argument the integrality assumption also implies that $(df)_x = (df)_0$ for every $x \in \mathbb{R}^m$.

Let $v \in \mathbb{R}^n$ be any vector. It can be decomposed into a sum of integer vectors v_j, $v = \sum a_j v_j$ with $a_j \in \mathbb{R}$. This allows to connect 0 and v with the broken path such that each of its segment is parallel to one of the integer vectors v_j. Therefore, we have

$$f(v) = \sum a_j (df)_0(v_j) = (df)_0(v).$$

$\square$

A map $f : M \to N$ is called an isomorphism (or a *symmetry*) of integer affine manifolds if it is invertible and both f and f^{-1} are morphism. Then we say that M and N are isomorphic as integer affine manifolds.

All isomorphisms of integer affine manifolds M form a group. If the quotient M/G by a subgroup G of this group is a manifold (which is the case if this subgroup acts in a properly discontinous fashion, i.e. every point x admits a neighborhood $U \ni x$ such that all translates by the elements of G are disjoint) then it gets a natural integer affine structure from M.

Clearly, $\mathbb{R}^n$ is an affine integer manifold tautologically. The group of its symmetries is the group of all integer affine-linear transformations of $\mathbb{R}^n$. The action of the whole group is not properly discontinous, so we need to restrict to a subgroup. The easiest properly discontinous subgroup is the lattice generated by translation in linearly independent directions. But there are other choices of subgroups, also using non-trivial linear parts (from $GL_n(\mathbb{Z})$).

Example 1.11. Consider the following examples of integer affine manifolds obtained as the quotients of $\mathbb{R}^2$. Let M be the quotient of the plane $\mathbb{R}^2$ by the subgroup Λ generated by the vectors $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}$.

For any choice of $a, b, c, d \in \mathbb{R}$ with $ad-bc \neq 0$ the resulting quotients are integer affine manifolds. All of them are diffeomorphic (and diffeomorphic to $S^1 \times S^1$). However they are not all isomorphic as integer affine manifolds.

E.g. if $b = 0$ then M is foliated by closed "horizontal" circles obtained as the quotient (t, s), where for each circle $s \in \mathbb{R}$ is fixed while $t \in \mathbb{R}$ varies. The condition $b = 0$ ensures that the points (t, s) and $(t + a, s)$ coincide so that we get a closed circle.

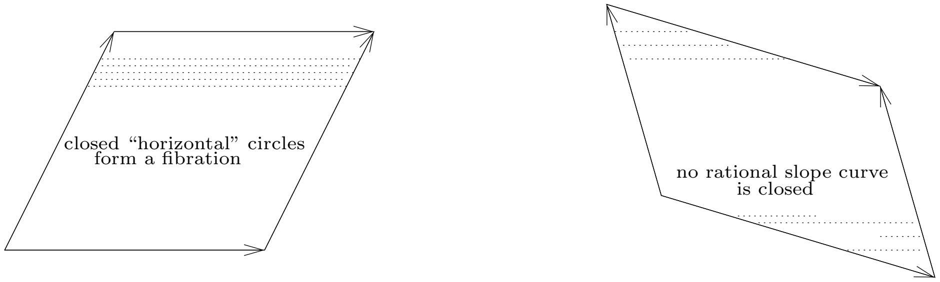

Figure 2. Different integer affine structures n $S^1 \times S^1$.

Of course, being "horizontal" is not an intrinsic condition in M and depends on the choice of chart to $\mathbb{R}^2$. But there is also an intrinsic property that holds for these circles.

Definition 1.12. Let $C \subset M$ be a curve. We say that it has *rational slope* if it is tangent to an integer vector at its every point.

Alternatively we may define such curves as those which have rational slope in each chart. This property does not depend on the choice of the charts while being "horizontal" in one chart ensures rational slope in others.

Definition 1.13. Let $C \subset M$ be a curve with rational slope and $v \in T_xC$ be a vector tangent to $x \in C$. We say that v is a *primitive vector* if it is integer in T_xM and cannot be presented as a non-trivial positive integer multiple of another integer tangent vector.

Proposition 1.14. *For any $x \in C$ the primitive tangent vector is unique up to sign.*

Proof. All vectors tangent to C form a 1-dimensional real vector space while the integer vectors form a lattice isomorphic to $\mathbb{Z} \subset \mathbb{R}$. $\square$

With the help of the primitive vectors we may define intrinsic *length* of a curve C with rational slope. Indeed, a 1-form α on C that takes value ± 1 on primitive vectors is unique up to sign. Let $\gamma \subset C$ be an arc on C.

Definition 1.15. The (intrinsic) length of γ is the integral $\int\limits_{\gamma} |\alpha|$.

In charts the intrinsic length can be obtained by taking the Euclidean length of γ and dividing it by the Euclidean length of a primitive vector parallel to γ.

We return to Example 1.11. If $b = 0$ we can measure the length of the "horizontal" circles. Clearly, all their lengths coincide and equal to $|a|$. If they are the only closed curves with rational slope on M then $|a|$ is the isomorphism invariant.

We may also choose $a, b, c, d \in \mathbb{R}$ linearly independent over $\mathbb{Q}$. Then no circle in M can have rational slope. Indeed, suppose that on the contrary we can find such a circle and it is parallel at its every point (in a chart obtained by reversing the quotient projection) to an integer vector $\begin{pmatrix} m \\ n \end{pmatrix} \in \mathbb{R}^2$. Then a multiple of $\begin{pmatrix} m \\ n \end{pmatrix}$ is proportional to an integer linear combination $j \begin{pmatrix} a \\ b \end{pmatrix} + k \begin{pmatrix} c \\ d \end{pmatrix}$, $j, k \in \mathbb{Z}$. But then $n(ja+kc) = m(jb+kd)$ which contradicts to the linear independence over $\mathbb{Q}$.

6. Examples of integer affine surfaces

Example 1.16. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the gliding reflection obtained by the composition of the reflection at the x-axis with a translation by $\begin{pmatrix} a \\ 0 \end{pmatrix}$, $a > 0$. Let B be a translation by $\begin{pmatrix} 0 \\ d \end{pmatrix}$, $d > 0$.

The quotient of $\mathbb{R}^2$ by the properly discontinous subgroup G of symmetries generated by R and B is a Klein bottle. Note that for this example we have foliations both by horizontal and vertical circles. All of them, except for two horizontal "core" circles have the same (intrinsic) length, equal to $2a$ and d respectively. The two horizontal "cores" have lengths equal to a, see Figure 3 for one of the "cores", the other is the result of identification of the horizontal sides of the rectangle.

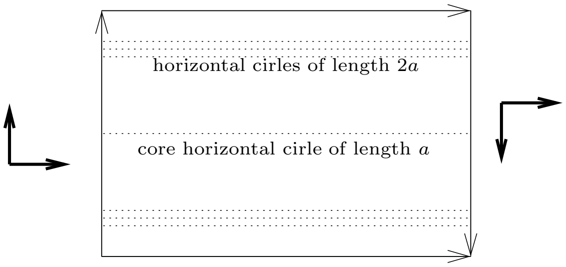

FIGURE 3. A Klein bottle with horizontal circles.

Note also that the vertical circles are dual to the first Stiefel-Whitney class. Therefore d is an isomorphism invariant of M. The manifold M can be obtained from the $a \times d$ rectangle by identifying the oriented opposite sides, cf. Figure 3. The conjugation of G by an element from $GL_2(\mathbb{Z})$ results in replacing the rectangle by a parallelogram, but the slopes of the sides of this parallelogram will still have rational slope. Of course, such conjugation does not change the isomorphism type of M.

So far our examples look very similar to examples of surfaces with Euclidean structure (cf. e.g. [**49**], [**67**]). Consider now a radically different example an integer affine structure on a torus. First, we construct a nontrivial integer affine annulus.

EXAMPLE 1.17. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a map obtained as the composition of a translation by $\begin{pmatrix} 0 \\ d \end{pmatrix}$, $d > 0$, and the linear transformation of $\mathbb{R}^2$ defined by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let R be the quotient of $\mathbb{R}^2$ by the group generated by A.

The surface R is an integer affine annulus that is different from the quotient of $\mathbb{R}^2$ by the group generated by any translation. Inside R we have immersed curves with rational slope that have self-intersections as shown on Figure 4. We identify the top side of the strip with the bottom so that the corresponding bases match.

Note that the shear transformation is the only possible linear part for an orientation-preserving deck transformation $\mathbb{R}^2 \to \mathbb{R}^2$ corresponding to an integer affine linear transformation as shown in the following proposition.

PROPOSITION 1.18. *If an integer affine linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ is fixed point free and orientation-preserving then its linear part L has both eigenvalues equal to 1.*

PROOF. Let λ and μ be the eigenvalues of L. Since $L \in SL_2(\mathbb{Z})$ and preserves orientation we have $\lambda\mu = 1$. If $\lambda, \mu \neq \mathbb{R}$ then $\lambda = \bar{\mu}$ as L is real. Therefore $|\lambda|^2 = |\mu^2| = 1$ and L is an orthogonal matrix in some basis so that A is a metric preserving transformation. From Euclidean planimetry we know that A must be a translation.

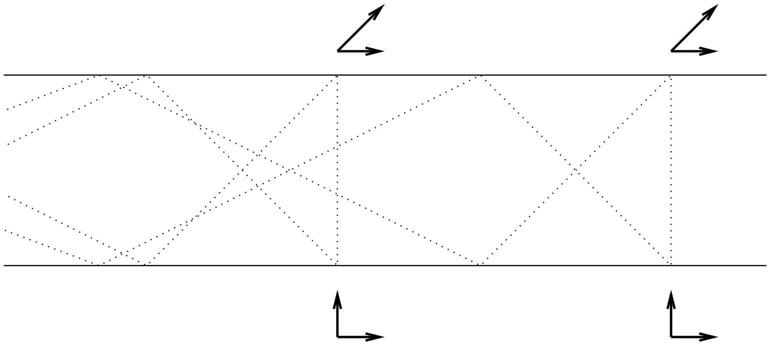

FIGURE 4. An integer affine annulus and two curves with rational slope there.

If $\lambda, \mu \in \mathbb{R}$ with $\mu = \frac{1}{\lambda} \neq \lambda$ then we may choose the coordinates in $\mathbb{R}^2$ so that L is given by $(x, y) \mapsto (\lambda x, \frac{1}{\lambda} y)$. Suppose that the translational part of A is given by $\begin{pmatrix} a \\ b \end{pmatrix}$. It suffices to find (x, y) such that $\lambda x - x = a$ and $\frac{1}{\lambda} y - y = b$. But these linear equations clearly have solutions if $\lambda \neq 0$. $\square$

EXAMPLE 1.19. Let T be the quotient of $\mathbb{R}^2$ by the group generated by the transformation A from Example 1.17 and a translation by $\begin{pmatrix} a \\ 0 \end{pmatrix}$. This is a compact surface diffeomorphic to the torus but not isomorphic to any quotient of $\mathbb{R}^2$ by a lattice of translations. Inside T we have immersed curves with rational slope that have self-intersections as in the case of Example 1.17.

EXAMPLE 1.20. Let K be the Klein bottle obtained as the quotient of $\mathbb{R}^2$ by the group generated by the transformation A from Example 1.17 and the transformation R from Example 1.16. This group also acts in a properly discontinuous manner so K is an integer affine surface. Just like the torus T from Example 1.19 the Klein bottle K has immersed curves with rational slope that have self-intersections.

In fact, Examples 1.19 and 1.20 admit the same tiling by fundamental domains in $\mathbb{R}^2$ shown in Figure 5.

REMARK 1.21. Any integer affine manifold can also be considered as a real affine manifold as we have embedding $GL_n(\mathbb{Z}) \subset GL_n(\mathbb{R})$. See [**5**], [**36**], [**47**] for a discussion of real affine structures, particularly on a torus. See also [**16**] for a discussion of affine structures with singularities.

7. Integer affine manifolds with corners

While integer affine manifolds are modeled on open sets in $\mathbb{R}^n$ the tropical affine space $\mathbb{T}^n$ has boundary and corners.

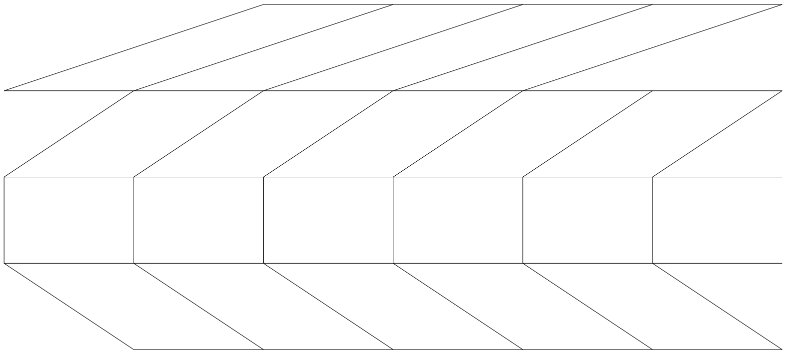

FIGURE 5. Tiling of $\mathbb{R}^2$ by fundamental domains for Examples 1.19 and 1.20.

DEFINITION 1.22. Let $x = (x_1, \dots, x_n)$ be a point in $\mathbb{T}^n = [-\infty, +\infty)^n$. We call the *sedentarity* $s(x)$ of x the number of coordinates x_j equal to $-\infty$.

The tropical affine space $\mathbb{T}^n$ is a manifold near its point x if and only if x has sedentarity 0.

Let $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ be an integer affine-linear map. Let L be the linear part of Φ which can be viewed as an (integer) $m \times n$ matrix. Let $x \in \mathbb{T}^n \smallsetminus \mathbb{R}^n$ be a point of positive sedentarity in $\mathbb{T}^n$. The image $\Phi(x)$ still makes sense as a point in $\mathbb{T}^m$ if whenever $x_j = -\infty$ the whole jth row of the matrix L is non-negative. Here we use the convention "$a(-\infty)$" $= -\infty$ if $a > 0$ and "$0(\infty)$" $= 0$.

This gives us partially-defined extensions

$$\bar{\Phi} : \mathbb{T}^n \dashrightarrow \mathbb{T}^m$$

of integer affine-linear maps $\mathbb{R}^n \to \mathbb{R}^m$. The map $\bar{\Phi}$ is continuous on the domain of its definition. We treat such maps as integer affine linear maps between affine tropical spaces. They allow us to extend the notion of integer affine structure to a larger class of spaces almost by repeating Definition 1.5

DEFINITION 1.23. Let X be a topological space. We say that X is an *integer affine manifold with corners* if X is enhanced with an open covering U_α and charts $\phi_\alpha : U_\alpha \to \mathbb{T}^n$ such that for each α, β the overlapping map $\phi_\beta \circ \phi_\alpha^{-1}$ can be obtained as the restrictions of a (partially defined) integer affine-linear map $\bar{\Phi}_{\beta\alpha} : \mathbb{T}^n \dashrightarrow \mathbb{T}^n$ that is defined everywhere of $\phi_\alpha(U_\alpha)$.

If $x \in \mathcal{U}_\alpha \subset X$ then we define its sedentarity as the sedentarity of its image $\phi_\alpha(x) \in \mathbb{T}^n$.

PROPOSITION 1.24. *The sedentarity $s(x)$ of a point $x \in X$ does not depend on the choice of the chart U_α.*

PROOF. Suppose, on the contrary, that $s(\bar{\Phi}_{\beta\alpha}(x)) < s(x) = k$ for $x \in \mathbb{T}^n$. Without the loss of generality we may assume that $x_1 = \cdots = x_k = -\infty$.

Then we know that the top k rows of the matrix giving the linear part L of $\bar{\Phi}_{\beta\alpha}$ must consist of non-negative (integer) numbers. Since L is invertible we may also assume without the loss of generality that the top $k \times k$ minor is not degenerate. But then the first k coordinates of $\bar{\Phi}_{\beta\alpha}(x)$ are all equal to $-\infty$ and this supplies a contradiction. $\square$

Let X_s be the locus of points of sedentarity s in an n-dimensional integer affine manifold with corners X.

Proposition 1.25. *The space X_s is a disjoint union of integer affine manifolds of dimension $n-s$ (without boundary or corners).*

Proof. Restrictions of the overlapping maps $\bar{\Phi}_{\beta\alpha}$ to the coordinate $(n-s)$-planes in $\mathbb{T}^n$ (those defined by $x_{j_1} = \dots = x_{j_s} = -\infty$) provides the required integer affine structure. $\square$

Definition 1.26. The integer affine structure on a manifold with corners is called *complete* if every component of X_s is a complete integer manifold for each $s = 0, \dots, n$.

Let X and Y be two integer affine manifolds with corners.

Definition 1.27. A map $f : X \to Y$ is called a *morphism* if for every $x \in X$ there exists charts $U^X_\alpha \ni x$, $U^Y_\beta \ni f(x)$ and a map $\Phi : \mathbb{T}^n \to \mathbb{T}^m$, $\mathbb{T}^n \supset U^X_\alpha$, $\mathbb{T}^m \supset U^Y_\beta$, such that $f(t) = (\phi^Y_\beta)^{-1} \circ \Phi \circ \phi^X_\alpha$.

Note that this is a straightforward extension of the definition of morphisms of manifolds without corners.

Clearly, any open subset U of a manifold with corner X is itself a manifold with corners (though not necessarily complete even in the case when the ambient manifold with corners X is complete).

Definition 1.28. The *(tropical) monomial* on U is any morphism $U \to \mathbb{T}$.

Proposition 1.29. *If U is complete then for any monomial $\kappa : U \to \mathbb{T}$ we have $\kappa(U) \supset \mathbb{R}$.*

Proof. The map κ can be lifted to a morphism from the universal covering $\tilde{U} \to \mathbb{T}$. Its image has to contain $\mathbb{R}$ as $\tilde{U}$ contains $\mathbb{R}^n$. $\square$

Note that $\mathbb{T}^n$ has the tautological structure of an integer affine manifold with corners. Furthermore, we can glue several copies of $\mathbb{T}^n$ together to get compact integer affine manifolds with corners. For us the most important example is that of the tropical projective space.

8. Tropical projective spaces

Consider the set

$$\mathbb{TP}^n = \mathbb{T}^{n+1} \smallsetminus \{0_{\mathbb{T}^{n+1}}\} / \sim$$

where $0_{\mathbb{T}^{n+1}} = (-\infty, \dots, -\infty)$ is the origin in $\mathbb{T}^{n+1}$ and we set $(x_0, \dots, x_n) \sim (y_0, \dots, y_n)$ if there exists $\lambda \in \mathbb{T}^\times$ such that $x_j =$ "λy_j" $= \lambda + y_j$ for any $j = 0, \dots, n$. Clearly the set $\mathbb{TP}^n$ gets a natural topology of the quotient. Furthermore, it admits a natural structure of an integer affine manifold with corners. As usual, we use the homogeneous coordinate notations $x = [x_0 : \dots : x_n] \in \mathbb{TP}^n$ to denote the equivalence class of $(x_0, \dots, x_n)$.

To see that we cover $\mathbb{TP}^n$ with $n+1$ open charts

$$U_j = \{x \in \mathbb{TP}^n \mid x_j \neq 0_{\mathbb{T}} = -\infty\},$$

$j = 0, \dots, n$, $(\phi_j(x))_k =$ "$\frac{x_k}{x_j}$" $= x_k - x_l$, $k \neq j$.Here $(\phi_j(x))_k$ denotes the kth coordinate of the image $\phi_j(x)$ and the target of ϕ_j is the hyperplane $\mathbb{T}^n \subset \mathbb{T}^{n+1}$ given by $\{x \in \mathbb{T}^{n+1} \mid x_j = 1_{\mathbb{T}} = 0\}$.

The overlapping maps $\bar{\Phi}_{jk} : \mathbb{T}^n \dashrightarrow \mathbb{T}^n$, $j \neq k$ are given by

$$(\bar{\Phi}_{jk})_l = \text{“}\frac{x_l x_k}{x_j}\text{”} = x_l + x_k - x_j.$$

Clearly $\bar{\Phi}_{jk}$ is an integer affine map defined on $\{x_j \neq -\infty\} \subset \mathbb{T}^n$.

PROPOSITION 1.30. *The space $\mathbb{TP}^n$ is homeomorphic to the n-simplex Σ_n so that a point inside a k-face of Σ_n corresponds to a point of sedentarity $n-k$. Furthermore, the integer affine structure induced in the interior of each k-face is isomorphic to the tautological integer affine structure on $\mathbb{R}^k$.*

PROOF. The map

$$x \mapsto \left(\frac{x_1}{|x_0| + \dots + |x_n|}, \dots, \frac{x_n}{|x_0| + \dots + |x_n|}\right)$$

provides the required homeomorphism to the standard simplex in $\mathbb{R}^n_{\geq 0}$ (cut by the half-space $x_1 + \dots + x_n \leq 1$). ☐

Similarly to Proposition 1.8 we get the following statement.

PROPOSITION 1.31. *If X and Y are integer affine manifolds with corners then $X \times Y$ is also an integer affine manifold with corners. Furthermore, $X \times Y$ is complete if and only if both X and Y are complete.*

REMARK 1.32. In a similar way we may construct tropical counterparts of more general toric varieties. A complex smooth toric variety is obtained by gluing several copies of affine spaces $\mathbb{C}^n$ (or, more generally, products of affine spaces $\mathbb{C}^k$ with tori $(\mathbb{C}^\times)^{n-k}$) by maps such that each coordinate is given by a monomial.

The tropical counterparts are obtained by gluing copies of $\mathbb{T}^k \times \mathbb{T}^{n-k}$ by the maps given by the corresponding tropical monomials. As in the case with projective space there is a sedentarity-preserving homeomorphism with the corresponding polyhedron (see e.g. [**15**]). E.g. Figure 6 shows the tropical plane blown up at 6 points which is diffeomorphic (as a manifold with corners) to a hexagon.

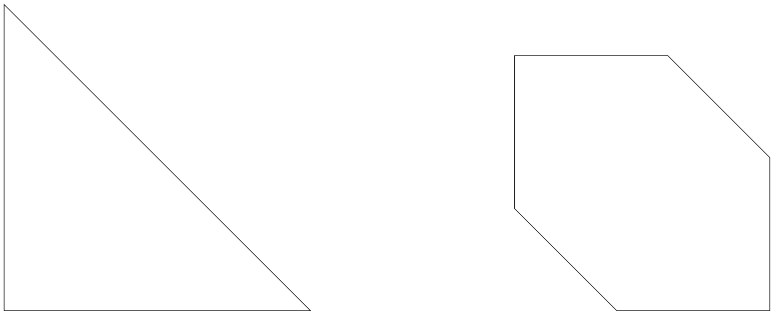

Figure 6. The tropical projective plane and the tropical projective plane blown up at three points. Interior of both polygons are isomorphic to the complete affine space $\mathbb{R}^2$ with the tautological integer affine structure.

CHAPTER 2

Some (semi-)algebraic notions

1. Tropical algebras

Definition 2.1. A $\mathbb{T}$-*cone* is a set V with a choice of an element $O \in V$ called *the origin* equipped with a *product* operation

$$\mathbb{T} \times V \to V,\ (a, v) \mapsto \text{“}av\text{”},$$

$a \in \mathbb{T}$, $v \in V$, such that “$(ab)v$” $=$ “$a(bv)$” for any $a, b \in \mathbb{T}$, $v \in V$, “av” $\neq$ “bv” if $a \neq b$ and “$v0_{\mathbb{T}}$” $= O$.

Definition 2.2. A *tropical algebra* A is a semiring (recall according to Definition 1.1 A has an additive zero $0_A \in A$ and a multiplicative unit $1_A \in A$) equipped with a $\mathbb{T}$-cone structure compatible with the semiring operations, i.e. such that “$a(fg)$” $=$ “$(af)g$” and $O = 0_A$, subject to the following additional property. For any $f, g, h \in A$ if “fg” $=$ “fh” for $f, g, h \in A$ then either we have equality $g = h$ or the element f is a zero divisor, i.e. there exists $\tilde{f} \in A$ such that “$f\tilde{f}$” $= 0_A$.

Proposition 2.3. *There is a natural embedding*

$$\iota_A : \mathbb{T} \subset A$$

which respects the semiring addition and multiplication: $\iota_A(\text{“}a + b\text{”}) = \text{“}\iota_A(a) + \iota_A(b)\text{”}$, $\iota_A(\text{“}ab\text{”}) = \text{“}\iota_A(a)\iota_A(b)\text{”}$, $\iota_A(-\infty) = 0_A$ *and* $\iota_A(0) = 1_B$.

Conversely if A is semiring and $\iota_A : \mathbb{T} \subset A$ is such an embedding then A is a tropical algebra as long as “$0_A f$” $= 0_A$ *and* “$\iota_A(a)f$” $\neq \iota_A(b)f$ *for any* $f \in A$ *and* $a \neq b \in \mathbb{T}$.

Implicitly using this proposition we identify $\mathbb{T}$ with its image in A. In particular, we have $0_A = -\infty \in A$ and $1_A = 0 \in A$.

Proof. Define $\iota_A(a) = \text{“}a1_A\text{”}$. Note that ι_A is an embedding since A is a cone. We have

$$\iota_A(\text{“}a + b\text{”}) = \text{“}(a + b)1_A\text{”} = \text{“}a1_A + b1_A\text{”} = \text{“}\iota_A(a) + \iota_A(b)\text{”}$$

and

$$\iota_A(\text{“}ab\text{”}) = \text{“}(ab)1_A\text{”} = \text{“}a(b1_A)\text{”} = \text{“}(a1_A)(b1_A)\text{”} = \text{“}\iota_A(a)\iota_A(b)\text{”}.$$

To check the converse statement we note that ι_A gives a $\mathbb{T}$-cone structure on A by “af” $=$ “$\iota_A(a)f$”. $\square$

Definition 2.4. Let A and B be two tropical algebras. A map $\phi : A \to B$ is called a *homomorphism* of tropical algebras (or just a $\mathbb{T}$-homomorphism) if for any $a, b \in A$ we have $\phi(\text{“}a+b\text{”}) = \text{“}\phi(a)+\phi(b)\text{”}$, $\phi(\text{“}ab\text{”}) = \text{“}\phi(a)\phi(b)\text{”}$ and, in addition, ϕ is identity on $\mathbb{T}$, i.e. the diagram

$$\begin{tikzcd}
\mathbb{T} \arrow[r, "\iota_A"] \arrow[dr, "\iota_B"'] & A \arrow[d, "\phi"] \\
 & B
\end{tikzcd}$$

is commutative. As usual, an *isomorphism* is an invertible homomorphism; an *epimorphism* is a surjective homomorphism and a *monomorphism* is an injective homomorphism.

Definition 2.5. A tropical algebra B is called an integral domain if it does not have zero divisors, i.e. for any $f, g \in B$ such that $\text{“}fg\text{”} = 0_B$ we have either $f = 0_B$ or $g = 0_B$.

2. Examples

Example 2.6. Consider the semiring

$$\mathbb{T}[x] = \{\text{“}\sum_{j=0}^{k} a_j x^j\text{”} \mid a_j \in \mathbb{T}, k \in \mathbb{N} \cup \{0\}\}$$

of formal tropical polynomials in one variable x. These polynomials can be added and multiplied according to formal polynomial laws (recall that $-\infty$ is our additive zero) and form The embedding $\iota : \mathbb{T} \subset \mathbb{T}[x]$ is tautological $a \mapsto a$.

Similarly, the semiring $\mathbb{T}[x_1, \dots, x_n]$ of formal tropical polynomials in n variables

$$\text{“}\sum_{(j_1,\dots,j_n)\in J} a_{j_1\dots j_n} x_1^{j_1} \dots x_n^{j_n}\text{”},$$

where $a_j \in \mathbb{T}$ and J is a finite subset of $(\mathbb{N} \cup \{0\})^n$, is another example of tropical algebra. For convenience we will use multi-index notations for multivariable monomials: if $x = (x_1, \dots, x_n) \in \mathbb{T}^n$ and $j = (j_1, \dots, j_n) \in \mathbb{Z}^n$ then

$$x^j = x_1^{j_1} \dots x_n^{j_n}.$$

Example 2.7. Consider the tropical algebra $\mathcal{O}(\mathbb{T}^n)$ of functions

$$\mathbb{T}^n \to \mathbb{T},\ x \mapsto f(x),$$

where $f \in \mathbb{T}[x_1, \dots, x_n]$ and $x = (x_1, \dots, x_n) \in \mathbb{T}^n$. The addition and multiplication on $\mathcal{O}(\mathbb{T}^n)$ are pointwise tropical addition and multiplication, while constant functions give the embedding $\mathbb{T} \to \mathcal{O}(\mathbb{T}^n)$. Elements of $\mathcal{O}(\mathbb{T}^n)$ are called *regular functions* on $\mathbb{T}^n$.

The tautological map

$$\tau : \mathbb{T}[x_1, \dots, x_n] \to \mathcal{O}(\mathbb{T}^n)$$

is an epimorphism. Note that $\tau^{-1}(-\infty) = \{-\infty\}$. Nevertheless, τ is not a monomorphism (unless $n = 0$). E.g.

$$\text{“}0x_1^2 + ax_1 + 0\text{”} = \text{“}0x_1^2 + 0\text{”}$$

whenever $a \le 0 \in \mathbb{T}$. Indeed, in this case we have (depending on $x_1 \in \mathbb{T}$) either “ax_1” $= x_1 + a \le 2x_1 =$ “$0x_1^2$” or “ax_1” ≤ 0.

Definition 2.8. A tropical algebra A is called *finitely generated* if there exist $f_1, \dots, f_n \in A$ such that any $f \in A$ can be presented in the form

$$f = \text{“}\sum_{j=1}^{n} a_j f_j\text{”}.$$

The elements $f_1, \dots, f_n$ are called *generators* of A.

Equivalently, A is finitely generated if there exists an epimorphism

$$\mathbb{T}[x_1, \dots, x_n] \to A$$

for some $n \in \mathbb{N}$.

Example 2.9. Consider the algebra $\mathbb{T}[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$ of Laurent polynomials in n variables “$\sum\limits_{j \in J} a_j x^j$”, where $a_j \in \mathbb{T}$ and J is a finite subset of $\mathbb{Z}^n$. This algebra is finitely generated by $2n$ generators $x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}$.

3. Spectra of tropical algebras

Let A be a tropical algebra. Let B_1 and B_2 be two other tropical algebras and $\phi_j : A \to B_j$ be two epimorphisms.

Definition 2.10. The maximal spectrum $\mathrm{Spec}_m(A)$ is the set of all $\mathbb{T}$-homomorphisms $A \to \mathbb{T}$.

The spectrum $\mathrm{Spec}(A)$ is the set of all epimorphisms $A \to B$ up to the equivalence above, where B is an integral domain.

Example 2.11. We have

$$\mathrm{Spec}_m(\mathbb{T}) = \mathrm{Spec}(\mathbb{T}) = \{\mathrm{pt}\},$$

the only tropical epimorphism of $\mathbb{T}$ to another tropical algebra is the identity $\mathbb{T} \to \mathbb{T}$.

Definition 2.12. If $f \in A$ and $x \in \mathrm{Spec}_m(A)$ then we define *the value* $f(x) \in \mathbb{T}$ as the image of f under the epimorphism $x : A \to \mathbb{T}$.

If $U \subset \mathrm{Spec}_m(A)$ we denote

$$\mathrm{Funct}(U) = \{g : U \to \mathbb{T} \mid \exists f \in A \;:\; \forall x \in U \; g(x) = f(x)\}.$$

Pointwise addition and multiplication turn $\mathrm{Funct}(U)$ to a tropical algebra. Clearly we have the natural *evaluation* epimorphism $A \to \mathrm{Funct}(U)$.

PROPOSITION 2.13. *For any* $a \in \mathbb{T}$ *and* $x \in \mathrm{Spec}_m(A)$ *we have*

$$\iota_A(a)(x) = a,$$

thus the image of ι_A *corresponds to the constant functions on* $\mathrm{Spec}_m(A)$.

PROOF. Since $x : A \to \mathbb{T}$ is a homomorphism of tropical algebras it is identity on $\mathbb{T}$. □

DEFINITION 2.14. The evaluation epimorphism $A \to \mathrm{Funct}(\mathrm{Spec}_m(A))$ is called the *reduction* epimorphism. We say that the tropical algebra A is *reduced* if the reduction epimorphism is an isomorphism.

Let $a : A \to B$ be a homomorphism of tropical algebras.

DEFINITION 2.15. The induced map

$$a^* : \mathrm{Spec}_m(B) \to \mathrm{Spec}_m(A)$$

is the map which takes an epimorphism $x : B \to \mathbb{T}$ to $x \circ a : A \to \mathbb{T}$.

Since a is an epimorphism of tropical algebras, so is $x \circ a$. In particular,it implies that $x \circ a$ maps onto $\mathbb{T}$.

PROPOSITION 2.16. *If* $a : A \to B$ *is an epimorphism of tropical algebras then* a^* *is an injection.*

PROOF. If $x \neq x' : B \to \mathbb{T}$ then there exists $f \in B$ such that $x(f) \neq x'(f)$. But then $x(a(g)) \neq x'(a(g))$ for any $g \in A$ such that $a(g) = f$. □

EXAMPLE 2.17. Any $x \in \mathrm{Spec}_m(A)$ is an epimorphism $A \to \mathbb{T}$. It induces an embedding $\mathrm{Spec}_m(\mathbb{T}) \subset \mathrm{Spec}_m(A)$ (cf. Example 2.11) that corresponds to the point x.

More generally, we have the following inclusions corresponding to such embeddings when we pass to considerations of the full spectrum $\mathrm{Spec}(A)$.

DEFINITION 2.18. If $x \in \mathrm{Spec}_m(A)$, $x : A \to \mathbb{T}$, and $F \in \mathrm{Spec}(A)$, $F : A \to B$, we say that x *is contained* in F if x is contained in the image $F^* : \mathrm{Spec}_m(B) \to \mathrm{Spec}_m(A)$. In other words, x is contained in F if there exists $y \in \mathrm{Spec}_m(B)$, $y : B \to \mathbb{T}$, such that $x = y \circ F$.

Thus F defines a subset of $\mathrm{Spec}_m(A)$. Clearly, this subset can be naturally identified with $\mathrm{Spec}_m(B)$.

DEFINITION 2.19. A subset $X \subset \mathrm{Spec}_m(A)$ is called *a basic closed set* if every tropical epimorphism $\mathrm{Funct}(X) \to \mathbb{T}$ corresponds to a point of X. In other words, if $x : \mathrm{Funct}(X) \to \mathbb{T}$ is a tropical epimorphism then the composition of the evaluation epimorphism $A \to \mathrm{Funct}(X)$ and x is contained in Z.

In other words X is closed if the evaluation epimorphism $A \to \mathrm{Funct}(X)$ defines X (and not a larger set).

PROPOSITION 2.20. *An intersection of basic closed sets in* $\mathrm{Spec}_m(A)$ *is a basic closed set.*

PROOF. Suppose that $X = \bigcap_j X_j$ and all $X_j \subset \mathrm{Spec}_m(A)$ are basic closed sets. Any tropical epimorphism $x : \mathrm{Funct}(X) \to \mathbb{T}$ can be composed with the restriction epimorphism $\mathrm{Funct}(X_j) \to \mathrm{Funct}(X)$. Therefore, the composition of the evaluation epimorphism $A \to \mathrm{Funct}(X)$ and x belongs to X_j for every j. □

Recall that any collection of subsets define a topology as a pre-basis. We apply this construction in the following definition.

DEFINITION 2.21. A set $X \subset \mathrm{Spec}_m(A)$ is called *closed* if it can be presented in the form

$$X = \bigcap_{\alpha \in J} X_\alpha,$$

where $J \ni \alpha$ is any parameterizing set and each X_α is the union of a finite number of basic closed sets.

It follows immediately from this definition that the intersection of any number of closed sets is closed and that the union of a finite number of closed sets is open as well. Furthermore, an empty set is closed as the parameterizing set J can be empty. The whole set $\mathrm{Spec}_m(A)$ is an example of a basic open set as it is presented by the identity epimorphism $A \to A$. Thus Definition 2.21 gives a topology on $\mathrm{Spec}_m(A)$.

A set $U \in \mathrm{Spec}_m(A)$ is called *open* if $\mathrm{Spec}_m(A) \setminus U$ is a closed set. We refer to this topology as the *spectrum* topology on $\mathrm{Spec}_m(A)$ to distinguish it from a different topology (the Zariski topology) which we introduce later on.

PROPOSITION 2.22. *The spectrum topology on* $\mathrm{Spec}_m(\mathbb{T}[x_1, \dots, x_n])$ *coincides with the Euclidean topology on* $[-\infty, +\infty)^n$*. Furthermore any closed set in the Euclidean topology is a basic closed set in the spectrum topology.*

PROOF. If $F \subset \mathrm{Spec}_m(\mathbb{T}[x_1, \dots, x_n])$ is a basic closed set then it corresponds to an epimorphism $\mathbb{T}[x_1, \dots, x_n] \to A$. Consider $\mathrm{Spec}_m(A)$. As each $\mathbb{T}$-homomorphism $A \to \mathbb{T}$ also gives a $\mathbb{T}$-homomorphism $\mathbb{T}[x_1, \dots, x_n] \to \mathbb{T}$ by composition we have the identification of $\mathrm{Spec}_m(A)$ and F. Since all tropical polynomials are continous functions any accumulation point of F also defines a $\mathbb{T}$-homomorphism $A \to \mathbb{T}$. Thus F must be closed.

Conversely, if $F \subset [-\infty, +\infty)^n$ is closed then we may consider the restriction homomorphism $\mathbb{T}[x_1, \dots, x_n] \to \mathrm{Funct}(F)$. If $y \notin F$ then we may have two tropical polynomials f, g such that $f(y) \neq g(y)$ but such that

$f(x) = g(x)$ for any $x \in F$. Thus a point $y \in [-\infty, +\infty)^n$ does not give a homomorphism from $\mathrm{Funct}(F)$ unless $y \in F$. $\square$

Similarly we get the following proposition.

PROPOSITION 2.23. *The spectrum topology on* $\mathrm{Spec}_m(\mathbb{T}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}])$ *coincides with the Euclidean topology on* $\mathbb{R}^n$. *Furthermore any closed set in the Euclidean topology is a basic closed set in the spectrum topology.*

4. Quotient semifields

As in classical Commutative Algebra if A is a tropical algebra A which is an integral domain then we can make a semifield $Q \supset \mathbb{T}$ out of T by allowing fractions.

LEMMA 2.24. *If* A *is a tropical integral domain and* $f_j, g_j \in A$, $g_j \neq 0_A$, $j = 1, 2, 3$, *are such that* “f_1g_2” = “f_2g_1” *and* “f_2g_3” = “f_3g”2 *then* “f_1g_3” = “f_3g_1”.

PROOF. Take a product of the left-hand and the right-hand sides of our hypotheses “f_1g_2” = “f_2g_1” and “f_2g_3” = “f_3g_2”. We get

$$“f_1g_2f_2g_3” = “f_2g_1f_3g_2”.$$

Since A is a tropical algebra either the statement of the lemma holds or “f_2g_2” is a zero divisor (cf. Definition 2.2). Since A is an integral domain and $g_2 \neq 0_A$ we have $f_2 = 0_A$. Then, in turn, $f_1 = 0_A$ and $f_3 = 0_A$ which also verifies the statement of the lemma. $\square$

DEFINITION 2.25. *The quotient semifield* $Q = \mathcal{R}\mathrm{at}(A)$ of a tropical integral domain A is the set of pairs (f, g), $f, g \in A$, $g \neq 0_A$ up to the following equivalence relation (cf. Lemma 2.24 $(f_1, g_1) \sim (f_2, g_2)$ if

$$f_1g_2 = f_2g_1 \in A.$$

We equip Q with operations of addition

$$“(f_1, g_1) + (f_2, g_2)” = (“f_1g_2 + f_2g_1”, “g_1g_2”)$$

and multiplication

$$“(f_1, g_1)(f_2, g_2)” = (“f_1g_1”, “f_2g_2”).$$

It is easy to see that the equivalence class of the results of these operations does not change if we replace (f_j, g_j), $j = 1, 2$, with an equivalent pair.

In accordance with the classical case we denote $(f, g) \in Q$ with “$\frac{f}{g}$”. Elements of the semifield Q are called *rational functions* associated with A.

From now on we suppose that a tropical algebra A is an integral domain and $Q = \mathcal{R}\mathrm{at}(A)$ is its quotient semifield.

PROPOSITION 2.26. *Q is a semifield that contains A as a subsemiring. The embedding $\mathbb{T} \subset A \subset Q$ makes Q into a tropical algebra. The map $q : A \to Q$, $q(f) =$ "$\frac{f}{1_A}$" is a monomorphism of tropical algebras.*

PROOF. Clearly, Q is a semiring since A is a semiring. Since we have the inversion operation

$$\text{“}\frac{1}{f/g}\text{”} = \text{“}\frac{g}{f}\text{”}$$

Q is a semifield. If "$\frac{a}{1_A}$" is equivalent to "$\frac{b}{1_A}$", $a, b \in A$ then, by definition, $a = b$. In particular, this gives an embedding $\mathbb{T} \subset Q$ which makes Q a tropical algebra and q a tropical algebra monomorphism. $\square$

PROPOSITION 2.27. *Any homomorphism $h : A \to B$ of tropical algebras naturally extends to a homomorphism $H : \mathcal{R}\mathrm{at}(A) \to \mathcal{R}\mathrm{at}(B)$.*

PROOF. We set $H(\text{“}\frac{f}{g}\text{”}) = \text{“}\frac{h(f)}{h(g)}\text{”}$. $\square$

The homomorphism q from Proposition 2.26 defines a map

$$q^* : \mathrm{Spec}_m(Q) \to \mathrm{Spec}_m(A)$$

by taking $x : Q \to \mathbb{T}$ to $x \circ q : A \to \mathbb{T}$.

DEFINITION 2.28. A point $x \in \mathrm{Spec}_m(A)$ is called *finite* if $x \in q^*(\mathrm{Spec}_m(A))$. We denote the set of all finite points in $\mathrm{Spec}_m(A)$ with $(\mathrm{Spec}_m(A))^\circ$.

PROPOSITION 2.29. *Non-zero elements of A have finite values at finite points of the spectrum. I.e. if $x \in \mathrm{Spec}_m(A)$ is finite and $f \neq -\infty \in A$ then $f(x) \neq -\infty \in \mathbb{T}$.*

PROOF. Since a homomorphism $x : A \to \mathbb{T}$ can be factorized through $q : A \to Q$ it can be extended to "$\frac{1_A}{f}$". We have "$\frac{1_A}{f}$"$(x) =$ "$\frac{1_\mathbb{T}}{f(x)}$" $\in \mathbb{T}$, therefore $f(x) \in \mathbb{T}^\times$. $\square$

EXAMPLE 2.30. Consider the tropical algebra $\mathbb{T}[x_1, \dots, x_n]$ from Example 2.7. Its quotient semifield coincides with the quotient semifield of the algebra $\mathbb{T}[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$ as "$\frac{x_j^{-1}}{1_\mathbb{T}}$" $\sim$ "$\frac{1_\mathbb{T}}{x_j}$" (recall that $1_\mathbb{T} = 0$). We denote the resulting semifield in these cases with $\mathbb{T}(x_1, \dots, x_n)$ and call its elements *tropical rational functions in n variables.*

PROPOSITION 2.31. *If $\Phi : A \to B$ is a homomorphism then*

$$\Phi^*((\mathrm{Spec}_m(B))^\circ) \subset (\mathrm{Spec}_m(A))^\circ.$$

PROOF. By Proposition 2.27 we have the induced map of the spectra of Q_A and Q_A $\mathrm{Spec}_m(B)^\circ \to \mathrm{Spec}_m(A)^\circ$ that agrees with Φ^* since H is an extension of h. The required map is induced by the composition $A \to Q_A \to Q_B$. $\square$

5. Affine and convex functions in a tropical algebra

Definition 2.32. An element f in a tropical algebra $A \supset \mathbb{T}$ is called a primitive affine function if $f \neq -\infty$ and whenever we have $f = a + b$, $a, b \in A$, and we have either $f = a$ or $f = b$.

Recall that $Q^* = Q \smallsetminus \{0\}$ is an abelian group with respect to tropical multiplication. Denote with $\mathcal{A}\mathrm{ff}(A)$ the subgroup of Q^* generated by all primitive affine functions in $A \subset Q$.

Definition 2.33. Elements of $\mathcal{A}\mathrm{ff}(A)$ are called *affine functions* associated with A.

An element of Q is called *convex* if it is a tropical sum of elements from $\mathcal{A}\mathrm{ff}(A) \subset Q$. All convex functions form a semiring $\mathcal{C}\mathrm{onv}(A) \subset Q$.

Proposition 2.34. *If $f \in \mathcal{A}\mathrm{ff}(A)$ and $a, b \in \mathcal{C}\mathrm{onv}(A)$ are such that $f = a + b$ then either $f = a$ or $f = b$.*

Proof. There exists a primitive affine functions $h \in A$ such that "$f + h$" $\in A$ is a primitive affine function while "$a + h$", "$b + h$" $\in A$. We have "$a + h + b + h$" $=$ "$a + b + h$" $=$ "$f + h$" which contradicts to the primitivity of "$f + h$". $\square$

Definition 2.35. We say that a tropical algebra A is *tame* if the following conditions hold:

- for every $c \in \mathbb{T}^*$ the image $\iota_A(c) \in A$ is a primitive affine function (we call such functions *constant*) so that $\mathbb{T}^* \subset \mathcal{A}\mathrm{ff}(A)$ is a subgroup;
- the quotient group $\mathcal{A}\mathrm{ff}(A)/\mathbb{T}^*$ is a free abelian group of finite rank;
- the subset $\mathcal{A}\mathrm{ff}(A)$ generates Q_A in the semifield sense.

Proposition 2.36. *If A is tame then for any $f \in Q_A$ there exist functions $g, h \in \mathcal{C}\mathrm{onv}(A)$ such that $f =$ "$\frac{g}{h}$".*

Proof. Since $\mathcal{A}\mathrm{ff}(A)$ provides a set of generators for the semifield Q_A any element in Q_A can be written as a ratio of two polynomial functions from the elements of $\mathcal{A}\mathrm{ff}(A)$. $\square$

Corollary 2.37. *If A is tame then* $\mathrm{Spec}_m(\mathcal{C}\mathrm{onv}(A)) = \mathrm{Spec}_m(Q)$.

Proof. Since we have the inclusion $\mathcal{C}\mathrm{onv}(A) \subset Q$ any epimorphism $Q \to \mathbb{T}$ determines an epimorphism $\mathcal{C}\mathrm{onv}(A) \to \mathbb{T}$ by taking restriction. Since $\mathcal{C}\mathrm{onv}(A)$ generates the semifield Q this gives an embedding $\mathrm{Spec}_m(Q) \subset \mathrm{Spec}_m(\mathcal{C}\mathrm{onv}(A))$.

To finish the proof we need to show that any epimorphism $x : \mathcal{C}\mathrm{onv}(A) \to \mathbb{T}$ can be extended to Q. This follows from Proposition 2.29 and Corollary 2.36. $\square$

Example 2.38. The free tropical algebra $A = \mathbb{T}[x_1, \dots, x_n]$ is tame. The group $\mathcal{A}\mathrm{ff}(A)$ corresponds to the group of all affine-linear functions

$f : \mathbb{R}^n \to \mathbb{R}$ whose slope is integer:

$$f(x) = < s, x > + t,$$

$s = (s_1, \dots, s_n) \in \mathbb{Z}^n$, $t \in \mathbb{R}$. The function f is primitive affine for A if $s_j \geq 0$, $j = 1, \dots, n$. Convex functions are finite tropical sums of elements of $\mathcal{A}\mathrm{ff}(A)$.

The tropical algebra $A' = \mathbb{T}[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$ is also tame. We have $A' \supset A$ and $\mathcal{A}\mathrm{ff}(A') = \mathcal{A}\mathrm{ff}(A) \subset A'$. All elements of $\mathcal{A}\mathrm{ff}(A')$ are primitive affine for A'.

The tropical semifield $\mathcal{R}\mathrm{at}(A) = \mathcal{R}\mathrm{at}(A')$ is itself a tropical algebra. However, it is not tame as $\mathcal{A}\mathrm{ff}(\mathcal{R}\mathrm{at}(A))$ is empty. E.g. both "$\frac{1_\mathbb{T}}{1_\mathbb{T}+x_1}$" and "$\frac{1_\mathbb{T}}{1_\mathbb{T}+x_1^{-1}}$" are elements of $\mathcal{R}\mathrm{at}(A)$. However, we have the following expression for the tropical sum of these elements

$$\text{“}\frac{1_\mathbb{T}}{1_\mathbb{T}+x_1} + \frac{1_\mathbb{T}}{1_\mathbb{T}+x_1^{-1}}\text{”} = \text{“}\frac{1_\mathbb{T}+x_1^{-1}+1_\mathbb{T}+x_1}{1_\mathbb{T}+x_1+x_1^{-1}+1_\mathbb{T}}\text{”} = \text{“}\frac{1_\mathbb{T}+x_1^{-1}+x_1}{1_\mathbb{T}+x_1+x_1^{-1}}\text{”} = 1_\mathbb{T}.$$

Thus $1_\mathbb{T}$ is not a primitive-affine function in Q.

6. Affine structure resulting from the semialgebraic data

If A is tame then $\mathcal{A}\mathrm{ff}(A)/\mathbb{T}^*$ is a free finitely generated Abelian group. Consider

$$T = \mathrm{Hom}(\mathcal{A}\mathrm{ff}(A)/\mathbb{T}^*, \mathbb{R}) \approx \mathbb{R}^n.$$

This is an affine space with the tautological integer affine structure.

PROPOSITION 2.39. *If A is tame then we have a natural embedding* $(\mathrm{Spec}_m)^\circ \hookrightarrow T$.

PROOF. The embedding $\mathcal{A}\mathrm{ff}(A) \subset A$ generates a homomorphism

$$\mathbb{T}[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}] \to Q_A. \tag{1}$$

This gives a map $(\mathrm{Spec}_m)^\circ \to T$. We need to show injectivity of this map.

Suppose that $s_1, s_2 \in (\mathrm{Spec}_m)^\circ$, $s_1, s_2 : Q_A \to \mathbb{T}$ are distinct, but they produce the same homomorphism after the composition with (1). But any element of Q_A can be expressed in terms of the elements from $\mathbb{T}[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$ (using addition, multiplication and division) since A is tame. As the values of the functions from $\mathbb{T}[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$ at s_1 and s_2 are all the same we get that the values of all the functions from Q_A at s_1 and s_2 are also the same which leads us to a contradiction. □

Thus we may treat the finite part of the maximal spectrum of a tame tropical algebra A as certain (sedentarity 0) points in the affine space associated to $\mathcal{A}\mathrm{ff}(A)$. This gives us a way to consider topological spaces much more general than integer affine manifolds with corners. Unfortunately, most of them won't be useful for us as they'll be rather far from being a manifold.

EXAMPLE 2.40. Let $K \subset [0,1] \subset \mathbb{R}$ be the Cantor set and let A be the space of all functions $A \to \mathbb{T}$ that can be obtained as a restriction of a tropical polynomial $f \in \mathbb{T}[x]$, $f : \mathbb{T} \to \mathbb{T}$, to K.

Then $\operatorname{Spec}_m(A) = K$ as we can evaluate any $f \in A$ on any point $y \in K$. Conversely, if $y : A \to \mathbb{T}$ is a $\mathbb{T}$-homomorphism then it gives a $\mathbb{T}$-homomorphism $\mathbb{T}[x] \to \mathbb{T}$ (as the restriction to $\mathbb{T}$ produces a $\mathbb{T}$-homomorphism $\mathbb{T}[x] \to A$) and thus corresponds to a point $y \in \mathbb{T}$. If $y \notin K$ then the homomorphism $y : \mathbb{T}[x] \to \mathbb{T}$ cannot factor through A as the value of f at y is not determined by the values at K.

Note that $\mathcal{A}\text{ff}(A) = \mathcal{A}\text{ff}(\mathbb{T}[x])$, so the tropical algebra A is still tame, so in a sense we are considering the Cantor set enhanced with an integer affine structure.

In the following chapters we introduce tropical n-dimensional varieties. Locally they may look like either $\mathbb{T}^n$ or some more general polyhedral n-dimensional complexes in $\mathbb{T}^N$, $N > n$. They will never look like the Cantor set from Example 2.40. The next example provides a tropical algebra whose spectrum is a tropical variety (as we'll see later).

EXAMPLE 2.41. Let A be the algebra obtained by restriction of tropical polynomials in two variables to the tripod $Y \subset \mathbb{T}^2$ defined by

$$Y = [(-\infty, 0), (0,0)] \cup [(0, -\infty), (0,0)] \cup [(0,0), (+\infty, +\infty)],$$

see Figure 1. The projection $(x, y) \mapsto x$ gives a map $\pi : Y \to \mathbb{T}$ that induces a homomorphism $\pi^* : \mathbb{T}[x] \to A$. Furthermore, the map $\sigma : \mathbb{T} \to Y$, $x \mapsto (x, \text{"}x + 0\text{"})$ also induces a homomorphism $\sigma^* : A \to \mathbb{T}[x]$ that is right inverse to π^*, i.e. $\pi^* \circ \sigma^* = \mathrm{Id}$. The map $\sigma^* \circ \pi^*$ gives a retraction of A to the subalgebra of functions constant on the ray $[(0, -\infty), (0,0)]$

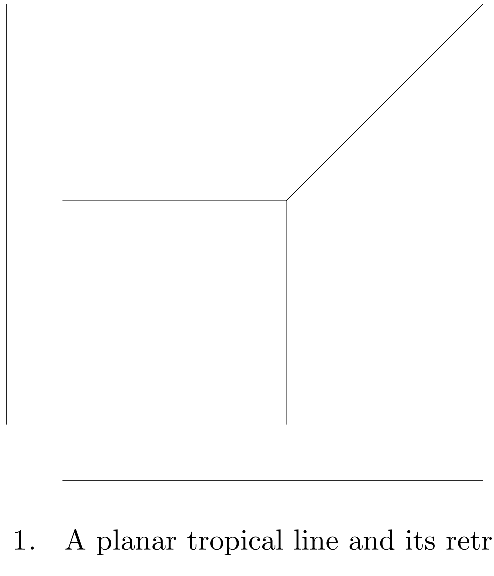

FIGURE 1. A planar tropical line and its retractions.

Note that Y is symmetric with respect to permutation of x and y. Thus we also have a right inverse to the projection homomorphism $\mathbb{T}[y] \to A$. We have $\operatorname{Spec}_m(A) = Y$, the space Y is called the planar tropical line.

7. Regular functions and tropical schemes

Let $f \in Q$ and $x \in \operatorname{Spec}_m(A)$. We say that *the value of f at x* is $f(x)$ if the epimorphism $x : A \to \mathbb{T}$ extends to an epimorphism $\bar{x} : \bar{A} \to \mathbb{T}$, where $\bar{A} \subset Q$ is a subalgebra such that $\bar{A} \supset A \cup \{f\}$ and $\bar{x}(f) = f(x)$. Since A generates Q as a semifield the value $f(x) \in \mathbb{T}$ is unique (if it exists). Note that for any $f \in Q$ and $x \in \operatorname{Spec}_m(A)^\circ$ the value $f(x)$ exists (and not equal to $-\infty \in \mathbb{T}$).

A point $x \in \operatorname{Spec}_m(A)$ is called *regular* for $f \in Q$ if there exists an open neighborhood $U \ni x$, $U \subset \operatorname{Spec}_m(A)$, and an element $g \in \mathcal{C}\mathrm{onv}(A)$ such that the values $g(y)$ and $f(y)$ exist and $g(y) = f(y)$ for any $y \in U$.

Let $U \subset \operatorname{Spec}_m(A)$ be any subset.

Definition 2.42. The tropical algebra $\tilde{\mathcal{O}}(U)$ *associated to a subset U* consists of all elements of Q that are regular at every point of U.

The tropical algebra $\mathcal{O}(U)$ consists of functions $f : U \to \mathbb{T}$ such that there exists an element $\tilde{f} \in \tilde{\mathcal{O}}(U) \subset Q$ such that $\tilde{f}(x) = f(x)$ for any $x \in U$. An element of $\mathcal{O}(U)$ is called *a regular function on U*.

Note that $\mathcal{O}(U)$ is a quotient of $\tilde{\mathcal{O}}(U)$ as we have the evaluation epimorphism

$$\mathrm{ev}^U_{\mathcal{O}} : \tilde{\mathcal{O}}(U) \to \mathcal{O}(U),$$

see Definition 2.14.

Definition 2.43. A point $x \in \operatorname{Spec}_m(A)$ is called a *pole* for $f \in Q$ if f is not regular at x. A point $x \in \operatorname{Spec}_m(A)$ is called a *zero* of $f \in Q$ if f is regular at x, but “$\frac{1}{f}$” has a pole at x.

Definition 2.44. Each element $f \in A$ defines a set $V_f \subset \operatorname{Spec}_m(A)$ of its zeroes. This set is called a hypersurface defined by f.

Proposition 2.45. *The union of finite number of hypersurfaces is a hypersurface.*

Proof. We claim that $\bigcup\limits_{j=1}^{n} V_{f_j}$ is a hypersurface defined by $\prod\limits_{j=1}^{n} f_j$. Clearly all points of $\operatorname{Spec}_m(A)$ are regular for any $f \in A$. Suppose that $x \in \operatorname{Spec}_m(A)$ is regular for “$\frac{1}{\prod_{j=1}^{n} f_j}$”. Then x is also regular for “$\frac{1}{f_j}$” as it can be obtained from “$\frac{1}{\prod_{j=1}^{n} f_j}$” by taking a product with all $f_{j'}$, $j' \neq j$. □

Definition 2.46. *A tropical scheme* is a pair consisting of a topological space X and a sheaf $\tilde{\mathcal{O}}_X$ of tropical algebras on X such that for every point $x \in X$ there is an open neighborhood $U \ni x$, a tropical integral domain A and an open set $U_A \subset \mathrm{Spec}_m(A)$ such that the pair $(U, \tilde{\mathcal{O}}_X|_U)$ is isomorphic to the pair $(U_A, \tilde{\mathcal{O}}_{\mathrm{Spec}_m(A)}|_{U_A})$.

The scheme is called reduced if for any U the tropical algebra $\tilde{\mathcal{O}}_X(U)$ is reduced. In such case we set $\mathcal{O}_X(U) = \tilde{\mathcal{O}}_X(U)$.

Since the restriction of a sheaf to an open set is a sheaf the tropical integral domain A has to be such that $\tilde{\mathcal{O}}_A|_{U_A}$ form a sheaf. Note that clearly we always have the required restriction homomorphisms $\rho_U^V : \tilde{\mathcal{O}}_A(V) \to \tilde{\mathcal{O}}_A(U)$ for $V \supset U$ that are also always monomorphisms as we just take an embedding of the elements of Q_A that are regular on V in the larger set of those elements which are regular on U. From now on we restrict our attention to reduced schemes X. The sheaf $\mathcal{O}_X$ is called *the structure sheaf of* X.

Let $f \in \mathcal{O}_X(V)$ and $x \in V$ for an open $V \subset X$ Choose an open neighborhood $U \ni x$, $U \subset \mathrm{Spec}_m(A)$. Thus x corresponds to a tropical epimorphism $x_A : A \to \mathbb{T}$. The *value of* $f(x)$ is $x_A(\rho_{U\cap V}^V(f)) \in \mathbb{T}$.

Proposition 2.47. *The value $f(x)$ does not depend on the choice of the affine neighborhood U.*

Proof. Suppose that x corresponds to a tropical epimorphism $x_{A'} : A' \to \mathbb{T}$ another affine neighborhood $U' \ni x$ with $U' = \mathrm{Spec}_m(A')$. Since $x \in U \cap U' \cap V$ both epimorphisms have to factor through the tropical algebra $\mathcal{O}(U \cap U' \cap V)$ where both $x_A(\rho_{U\cap V}^V(f))$ and $x_{A'}(\rho_{U'\cap V}^V(f))$ have a common lift $\rho_{U\cap U'\cap V}^V(f)$. □

Definition 2.48. Let $Z \subset X$ be any subset and $f : Z \to \mathbb{T}$ be a function. The function f is called *regular* if for any $x \in Z$ there exists an open neighborhood $U \ni x$ and $g \in \mathcal{O}_X(U)$ such that $f(y) = g(y)$ for any $y \in Z \cap U$.

Once again, all regular functions on Z together with pointwise addition and multiplication form a tropical algebra which we denote $\mathrm{Funct}(Z)$.

8. Regular maps

Definition 2.49. A *regular map* between tropical schemes

$$\Phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

is a pair consisting of a continuous map

$$f : X \to Y$$

and a collection of tropical algebra homomorphisms

$$\Phi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\Phi^{-1}(U))$$

for any open set $U \subset Y$ that is consistent with the restriction homomorphisms of the sheaves $\mathcal{O}_X$ and $\mathcal{O}_Y$, i.e. such that for any pair of open sets $V \subset U \subset Y$ the diagram

$$\begin{tikzcd}
\mathcal{O}_Y(U) \arrow[r, "\rho_V^U"] \arrow[d, "\Phi^*"] & \mathcal{O}_Y(V) \arrow[d, "\Phi^*"] \\
\mathcal{O}_X(\Phi^{-1}(U)) \arrow[r, "\rho_{\Phi^{-1}(V)}^{\Phi^{-1}(U)}"] & \mathcal{O}_Y(\Phi^{-1}(V))
\end{tikzcd}$$

is commutative. Here ρ_V^U are the corresponding restriction homomorphisms for regular functions.

For simplicity of notations we will often suppress the symbols $\mathcal{O}_X$ and $\mathcal{O}_Y$ and write a regular map just as $\Phi : X \to Y$.

Definition 2.50. A regular map $\Phi : X \to Y$ is called *a scheme embedding* if Φ is a set-theoretical embedding and for all open $U \subset Y$ the homomorphisms $\Phi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\Phi^{-1}(U))$ is an epimorphism. In this case X is called *a closed subscheme* of Y, once we identify X with $\Phi(X) \subset Y$.

Let $V \subset Y$ be any set and $W = \Phi^{-1}(V)$. Suppose that $f \in \text{Funct}(V)$ and $\Phi : X \to Y$ is a regular map. As usual, we have a set-theoretical pullback of the function f, namely $\Phi^*(f) : W \to \mathbb{T}$, $x \mapsto f(\Phi(x))$.

Proposition 2.51. *The function $\Phi^*(f)$ is regular in U, i.e. $\Phi^*(f) \in \text{Funct}(W)$.*

Proof. Since $f \in \text{Funct}(V)$ for every $x \in V$ there exists an open neighborhood $U \ni x$ and $g \in \mathcal{O}_Y(U)$ such that $g(y) = f(y)$ for every $y \in U \cap V$. We have $\Phi^*(g) \in \mathcal{O}_X(\Phi^{-1}(U))$ by definition of the tropical map and, clearly, $\Phi^*(g)(z) = f(\Phi(z))$ for every $z \in \Phi^{-1}(U) \cap W$. $\square$

CHAPTER 3

Hypersurfaces and complete intersections in $\mathbb{T}^n$

1. Integer affine manifolds as tropical schemes

After a bit of algebraic formalism we return to our geometric objects: integer affine manifolds.

THEOREM 3.1. *Any integer affine manifold X with corners can be naturally considered as a reduced tropical scheme.*

PROOF. Locally X is modeled on an open set in $\mathbb{T}^n = \mathrm{Spec}_m(\mathbb{T}[x_1, \dots, x_n])$ so that regular functions correspond to monomials, cf. Definition 1.28. $\square$

In particular, we may characterize the regular functions in terms of integer-affine structure. Recall that a monomial is just a affine-linear morphism to $\mathbb{T}$.

Let $U \subset X$ be an open set and $f : U \to \mathbb{T}$ be a continuous function.

PROPOSITION 3.2. *A function $f : U \to \mathbb{T}$ is regular at $x \in U$ if and only if there exist an open subset $W \subset U$ and a finite collection of monomials $\kappa_1, \dots, \kappa_l : W \to \mathbb{T}$ such that $f|_W = \max\{\kappa_1, \dots, \kappa_l\}$.*

PROOF. We may choose W so that it is contained in a single chart $\phi_\alpha : U_\alpha \to \mathbb{T}^n$. Then the second characterization coincides with the definition of a tropical polynomial. $\square$

Also we may speak about tropical hypersurfaces in integer affine manifolds with corners. A subspace $V \subset X$ is called a hypersurface if for any $x \in V$ there exists a chart $\phi_\alpha : U_\alpha \to \mathbb{T}^n$ and a tropical polynomial $f_\alpha : \mathbb{T}^n \to \mathbb{T}$. such that $V \cap U_\alpha = \phi_\alpha^{-1}(V_{f_\alpha})$, where V_{f_α} is the hypersurface associated to f_α. Thus to see the structure of hypersurfaces in X it suffices to look carefully at the structure of hypersurfaces in $\mathbb{T}^n$.

2. Hypersurfaces in $\mathbb{T}^n$

Let $f : \mathbb{T}^n \to \mathbb{T}$ be a tropical polynomial

$$f(x) = \text{“}\sum_{j \in \mathbb{Z}^n} a_j \kappa_j(x)\text{”} = \max_j a_j + \kappa(x), \tag{2}$$

$x = (x_1, \dots, x_n) \in \mathbb{T}^n$. Here the some is taken over the finite number of multi-indexes j, $a_j \in \mathbb{T}$ and $\kappa_j(x) = \text{“}x_1^{j_1} \dots x_n^{j_n}\text{”}$, so that “$a_j \kappa_j$” are monomials. Recall that the hypersurface V_f is the locus of all points $x \in \mathbb{T}^n$ such that "$\frac{1_{\mathbb{T}}}{f}$" is not regular at x.

Proposition 3.3. *The hypersurface* V_f *is the locus of points* $x \in \mathbb{T}^n$ *where the maximal value in* (2) *is attained by more than one monomial* $a_j \kappa_j$.

Proof. If more than one monomial assumes the maximum at x then f is strictly convex at x and thus "$\frac{1_{\mathbb{T}}}{f}$" $= -f$ cannot be convex. If only one monomial is maximal at x then f is locally linear at x and thus $-f$ is also regular at x. $\square$

The monomials $a_j \kappa_j$ naturally define a stratification of V_f. Let $J = \{j \in \mathbb{Z}^n \mid a_j \neq 0_{\mathbb{T}}\}$ be the indices parameterizing the monomials that appear in f. The set J is finite since f is a polynomial. For each $x \in \mathbb{T}^n$ we define

$$K_f(x) = \{j \in J \mid f(x) = \text{“}a_j \kappa_j\text{”},$$

in other words $K_f(x)$ is the set of the indices of the monomials where $f(x)$ assumes its maximum. Vice versa, for a subset $K \subset J$ of cardinality greater than one we may define the stratum $V_f^K \subset V_f$ by

$$V_f^K = \{x \in \mathbb{T}^n \mid K_f(X) = K.$$

Note that V_f^K is defined by a system of linear inequalities in $\mathbb{T}^n \supset \mathbb{R}^n$ and thus is a convex polyhedron (possibly unbounded) in $\mathbb{T}^n$. This means that it is the closure in $\mathbb{T}^n$ of a convex polyhedral domain in $\mathbb{R}^n$.

Proposition 3.4. *We have* $\mathbb{T}^n = \bigcup\limits_{K \subset J} V_f^K$ *and*

$$V_f = \bigcup_{|K|>1} V_f^K.$$

Each component of $\mathbb{T}^n \setminus V_f$ *naturally corresponds to a point* $j \in J$, *such that* "$a_j \kappa_j$" *is maximal in this component.*

Proof. this proposition is the direct corollary of Proposition 3.3. $\square$

For many subsets of J we have $V_f^K = \emptyset$. If $V_f^K = \emptyset$ we say that $K \in \mathrm{Subdiv}_f$ and denote with Δ_K the convex hull of K in $\mathbb{R}^n \supset K$. Denote with Δ_f the *Newton polyhedron* of f, i.e. the convex hull of J in $\mathbb{R}^n$. Each Δ_K is contained in a minimal affine-linear subspace in $\mathbb{R}^n$. Denote with Δ_K° the *relative interior* of Δ_K, i.e. the interior in the corresponding affine-linear space.

Theorem 3.5. *The polyhedra* Δ_K *form a subdivision of the polyhedron* Δ_f *which is dual to the corresponding strata* V_f^K. *Namely, we have the following properties.*

- *If* $K_1, K_2 \in \mathrm{Subdiv}_f$ *and* $K_1 \cap K_2 \neq \emptyset$ *then* $K_1 \cap K_2 \in \mathrm{Subdiv}_f$ *and* $\Delta_{K_1} \cap \Delta_{K_2} = \Delta_{K_1 \cap K_2}$.

- *The (relatively) open polyhedra* Δ°_K *are disjoint: for any* $K_1, K_2 \in \mathrm{Subdiv}_f$, $K_1 \neq K_2$ *we have* $\Delta^\circ_{K_1} \cap \Delta^\circ_{K_2} = \emptyset$.
- $\Delta_f = \bigcup\limits_{K \in \mathrm{Subdiv}_f} \Delta^\circ_K$.
- *For any* $K \in \mathrm{Subdiv}_f$ *we have* $\dim V^K_f + \dim \Delta_K = n$. *Furthermore the affine-linear subspaces in* $\mathbb{R}^n$ *generated by* $V^K_f \cap \mathbb{R}^n$ *and* Δ_K *are orthogonal. (More rigorously, the Newton polygon* Δ_f *and the hypersurface* $V_f \cap \mathbb{R}^n$ *belong to dual vector spaces* $\mathbb{R}^n$*, but we may identify them by introducing a scalar product to* $\mathbb{R}^n$*.)*
- *If* $\Delta_{K_1} \subset \Delta_{K_2}$ *then* $V^{K_1}_f \supset V^{K_2}_f$.

In particular, to each facet (i.e. an $(n-1)$*-dimensional face of* V_f*) we may associate a positive integer number equal to the* integer length *of the corresponding interval in* Subdiv_f. *Here the integer length of an interval* $I \subset \mathbb{R}^n$ *with* $\partial I \in \mathbb{Z}^n$ *is the total number of integer subintervals in it (i.e.* $\#(I \cap \mathbb{Z}^n) - 1$*).*

Proof. The last two properties come as straightforward applications of Linear Algebra.

Note that for every $j \in J$ the locus "$a_j \kappa_j(x)$" $= f(x)$ is defined with a system of linear inequalities and therefore is convex. Suppose that $K \in \mathrm{Subdiv}_f$ and $k \in \Delta^\circ_K$. Then, by convexity, "$a_k \kappa_j(k)$" $= f(x)$ exactly on V^K_f. Thus without loss of generality we may assume that K coincides with $\Delta_K \cap \mathbb{Z}^n$.

Thus Δ°_K are disjoint and form a subdivision of Δ_f. Suppose that $K_1 \cap K_2 \neq \emptyset$. Then a generic point x of the convex hull of $V^{K_1}_f \cup V^{K_2}_f$ must correspond to $K_1 \cup K_2$. □

Remark 3.6. Subdivisions that appear in Theorem 3.5 are called convex, regular or, sometimes, coherent lattice subdivisions of the polyhedron Δ_f, cf. e.g. [**15**]. The function $j \mapsto a_j$ is called the *height function* of the subdivision. In real algebraic geometry such subdivisions appeared after the discovery of the patchworking technique by Viro [**68**]. These subdivisions come as projections of the top faces of the polyhedral domain in $\mathbb{R}^n \times \mathbb{R}$ obtained as the convex hull of the undergraph of $j \mapsto a_j$, see [**15**].

Not all subdivisions are convex. Figure 1 depicts a classical example of a non-convex lattice subdivision (see e.g. [**68**], [**15**]). To see non-existence of the height function it suffices to look at the attachments of the would-be faces around the inner square.

Remark 3.7. Theorem 3.5 gave a description of hypersurfaces in $\mathbb{T}^n$. However, the same construction works also for hypersurfaces V in $(\mathbb{T}^\times)^n$, $\mathbb{TP}^n$ and other toric varieties as long as every component of V (its subset that constitute a hypersurface itself) has non-empty intersection with the torus $(\mathbb{T}^\times)^n$. Then the hypersurface V is still given by a tropical polynomial f in

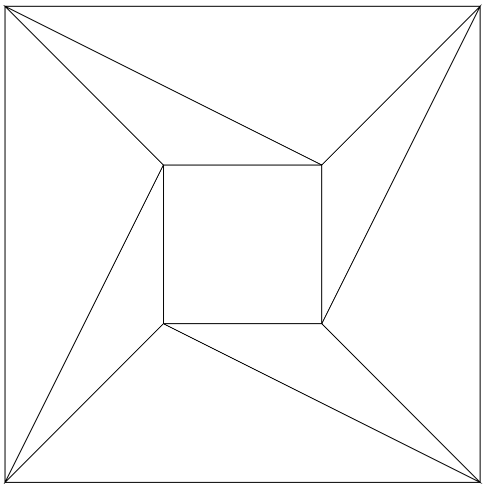

FIGURE 1. A non-convex lattice subdivision.

n variables and can be obtained by taking the closure in the corresponding toric variety of the toric part $V_f \cap (\mathbb{T}^\times)^n$ of the affine hypersurface $V_f \subset \mathbb{T}^n$. We often will use the same notation V_f for a hypersurface in other toric varieties.

3. Lines in the plane

The easiest examples to visualize are planar curves, i.e. hypersurfaces in $\mathbb{T}^2$. Note that Y from Example 2.41 is an example of a tropical line in the plane. Indeed, it is the hypersurface of $f(x, y) =$ “$x + y + 1_\mathbb{T}$”. All three monomials are equal at the origin while everywhere on the three rays two of the three monomials are equal, but greater then the third remaining monomial.

A general polynomial of degree 1 in two variables is of the form

$$f(x, y) = \text{“}ax + by + c\text{”}.$$

Thus a line in $\mathbb{T}^2$ is the hypersurface associated to this tropical polynomial. Note that as long as $a, b, c \neq 0_\mathbb{T}$ any tropical line can be obtained from Y by a translation in $\mathbb{R}^2$. More precisely, we have to take $Y \cap \mathbb{R}^2$, apply the translation and take the closure in $\mathbb{T}^2$ again.

Indeed, the hypersurface, associated to “$\frac{f(x,y)}{c}''= f(x, y) - c = \max\{(x+a-c, y+b-c, 0\}$ coincides with V_f. But $\max\{(x+a-c, y+b-c, 0\}$ corresponds to $\max\{x, y, 0\}$ under the translational change of coordinates $x \mapsto x + a - c$, $y \mapsto y + b - c$. Note that the horizontal and vertical rays of Y end with an infinite point (as the axes $\{y = -\infty\}$ and $\{x = -\infty\}$ are included in $\mathbb{T}^2$), but the diagonal ray is open.

If one of the coefficients of f assumes the value $0_{\mathbb{T}} = -\infty$ then the corresponding monomial is never maximal in f. Thus the corresponding hypersurface is the closure of the straight line which maybe horizontal, vertical or diagonal, depending on which monomial disappears, see Figure 2.

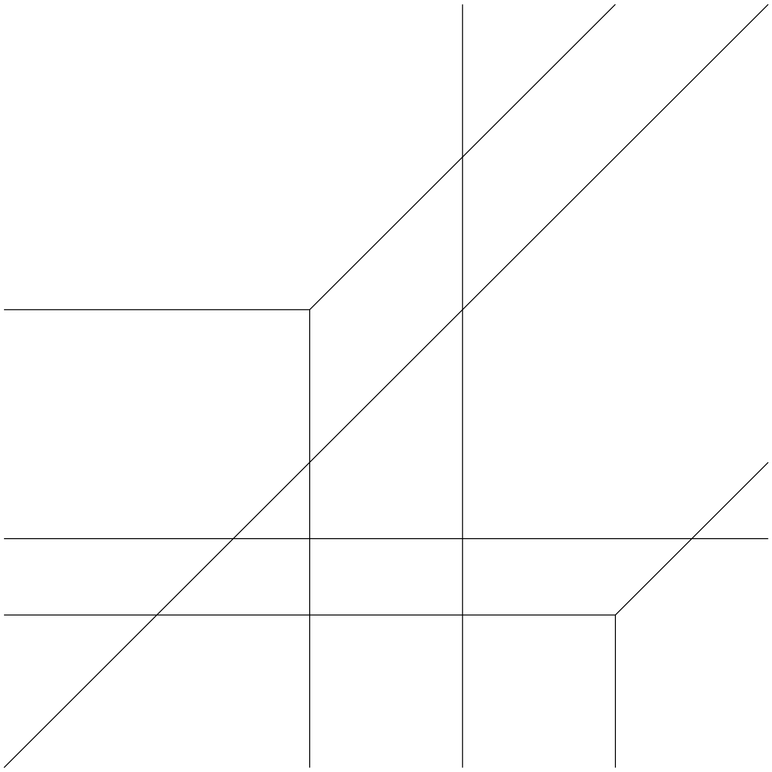

FIGURE 2. Five lines in $\mathbb{T}^2$.

Consider now the case when two of the coefficients of f assume the value $-\infty$. If $f(x,y) = c$, $c \in \mathbb{T}^\times$, then "$\frac{1_{\mathbb{T}}}{f}$" $= -c$ is regular everywhere on $\mathbb{T}^2$, so $V_f = \emptyset$. If $f(x,y) =$ "ax" $= x + a$, $a \in \mathbb{T}^\times$, then "$\frac{1_{\mathbb{T}}}{f}$" $= -x - a$ is regular as long as $x \neq infty$, but not defined at the coordinate y-axis $\{x = -\infty\}$ of $\mathbb{T}^2$. Thus in this case V_f coincides with the y-axis. Similarly the hypersurface of $f(x,y) =$ "ax" $= x + b$, $b \in \mathbb{T}^\times$, is the x-axis of $\mathbb{T}^2$.

The projective space $\mathbb{TP}^2$ provides a compactification of $\mathbb{T}^2$ by attaching an extra line (called the infinite line). When we consider, e.g. a family

$$f_t = \text{“}ty + c\text{”},\ t \to -\infty \tag{3}$$

the corresponding horizontal line moves to infinity and coincides with that infinite line in the limit.

We may draw the corresponding deformation on the (finite) triangle. For that we need to reparameterize $\mathbb{R}^2$ to the interior of a finite triangle.

One of the most natural ways (along with the map provided by Proposition 1.30 to do that is via the combination of the logarithmic moment map

$$\mathrm{Log} : (\mathbb{C}^\times)^2 \to \mathbb{R}^2,\ \mathrm{Log}(z, w) = (\log |z|, \log |w|),$$

which is the moment map for the $(\mathbb{C}^\times)^2$-invariant form $\frac{dz}{z} \wedge \frac{d\bar{z}}{\bar{z}} + \frac{dw}{w} \wedge \frac{d\bar{w}}{\bar{w}}$ and the Fubini-Study moment map for $\mathbb{CP}^2$

$$\mu : (\mathbb{C}^\times)^2 \to \mathbb{R}^2,\ \mu(z, w) = (\frac{|z|^2}{1 + |z|^2 + |w|^2}, \frac{|w|^2}{1 + |z|^2 + |w|^2}).$$

Note that the image $\mu(\mathbb{R}^2)$ is the interior of the triangle $T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ x + y \leq 1\}$. Both maps Log and μ have the same fibers, so we have a well-defined map $\mu \circ \mathrm{Log}^{-1} : \mathbb{R}^2 \to \mathrm{Int}(T)$, which is a diffeomorphism. Furthermore, this diffeomorphism extends to a diffeomorphism $\mathbb{TP}^2 \to T$. When we need to speak about the infinite points of varieties in $\mathbb{TP}^2$ it is more convenient to draw their images under this reparameterization. Note though that the image of a straight line in $\mathbb{R}^2$ is (in general) no longer straight in T.

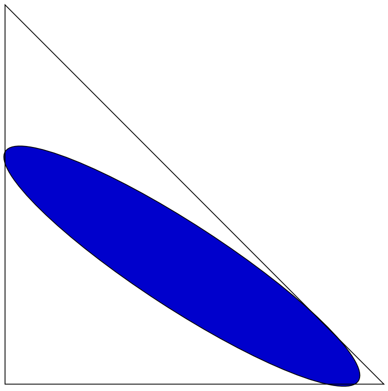

FIGURE 3. The image of a complex projective lines under $\mu \circ \mathrm{Log}^{-1}$ is an inscribed ellipse in T.

REMARK 3.8. one of the advantages of the parameterization $\mu \circ \mathrm{Log}^{-1}$ with respect to the parameterization provided by Proposition 1.30 is that the image of a line in $\mathbb{RP}^2$ is an ellipse that is tangent to the three sides of the triangle T, see Figure 3. The points of tangency with the sides corresponds to the points of intersection with the three coordinate axes (the x-axis, the y-axis and the infinite line). These tangencies divide the circle into three arcs, each corresponding to the real points of a line in a quadrant of $(\mathbb{R}^\times)^2$. Note that a generic line in $\mathbb{R}^2$ intersects three out of four quadrants.

The imaginary points of a line $L \subset \mathbb{CP}^2$ that is real (i.e. invariant with respect to the complex conjugation) are mapped inside this ellipse in the

2-1 fashion so that the the inverse image of a point inside the ellipse under $\mathrm{Log}|_L$ consists of a pair of complex conjugate points.

Furthermore, the image of any (not necessarily real) line in $\mathbb{CP}^2$ is the region in $\mathbb{T}^2$ that is encompassed by an ellipse tangent to the sides of T. Indeed, any line in $\mathbb{CP}^2$ can be made real after the multiplication in $(\mathbb{C}^\times)^2$ by a suitable pair (a, b), $a, b \in \mathbb{C}^\times$. Note that the family of ellipses in $\mathbb{R}^2$ is 5-dimensional and each tangency gives a condition of codimension 1. Thus we have a 2-dimensional family of suitable ellipses and this corresponds to the dimension of the space of lines in the projective plane.

The lines given by a binomial equation pass through an intersection point of the coordinate axes (recall that we treat the infinite line as one of the coordinate axes!) and correspond to the degeneration of ellipses to intervals passing through a vertex of the triangle. The lines given by a monomial coincide with one of the coordinate axes and correspond to a side of the triangle.

The same parameterization works well for images of tropical lines. Indeed, a generic line is made of three segment, where each segment is a subinterval of a line passing through a vertex of T, see the first part of Figure 4. The second part of this figure shows how generic lines degenerate to a binomial line. The last part of Figure 4 depicts the family (3) and its limit.

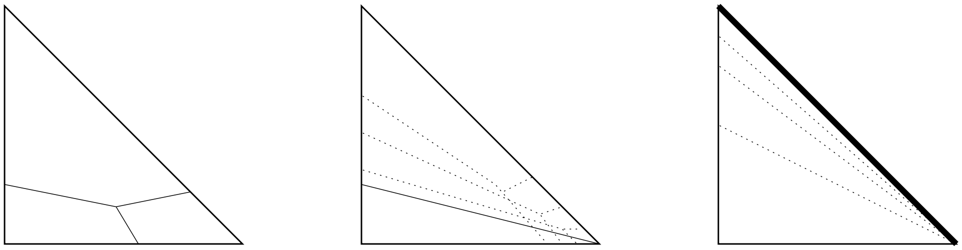

FIGURE 4. Images of tropical lines in T and their degenerations.

Thus we see that any line in $\mathbb{TP}^2$ is either an $\mathbb{R}^2$-translate of the tripod Y from Example 1 or a degeneration of such translates. Note that two generic lines in $\mathbb{TP}^2$ intersect in a unique point: e.g. any pair of lines in Figure 2 has such "transverse" intersection. In the same time we may find two lines that have a whole ray in common, see Figure 5.

Later in this book we develop the tropical intersection theory which allows to associate the cycle of the right dimension even for non-transverse cycles A, B. This intersection cycle will be supported on the skeleton of the set-theoretical intersection of the expected dimension. Each facet F of this skeleton will be included to the cycle with an integer (possibly negative) coefficient that depends only on the local structure of A and B near E.

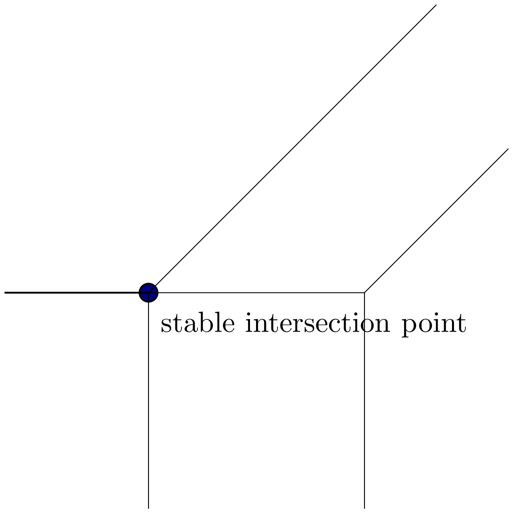

FIGURE 5. Non-transverse intersection of two lines in $\mathbb{T}^2$.

In particular, even though the lines from Figure 5 intersect along a ray, their intersection cycle is the (sedentarity 0) endpoint of this ray. This agrees with the notion of the *stable intersection* from [**56**] in the case when the ambient space is an (integer affine) smooth variety.

PROPOSITION 3.9. *Any pair of points $p_1, p_2 \in \mathbb{TP}^2$ can be joined with a line. Furthermore, this line is unique unless this pair of line and one of the intersection points of the coordinate axes (the points of sedentarity 2) are collinear.*

PROOF. Applying a translation in $\mathbb{R}^2$ to the tripod Y from Example 1 we may find a line $L \in \mathbb{TP}^2$ such that its 3-valent vertices coincide with p_1. If the sedentarity of p_1 is positive then we may find a line $L \ni p_1$ and a family of non-degenerate lines L_t so that the the trivalent point of L_t tends to p_1. Generically, the line L separates $\mathbb{TP}^2$ into three sectors, see Figure 6. If $p_2 \notin L$ then it is inside one of these sector. We can move L into this sector so that p_2 is remained on L by a translation antiparallel to the ray opposite to the sector of p_2. $\square$

THEOREM 3.10. *Lines in $\mathbb{TP}^2$ form themselves an integer affine manifold with corners isomorphic to $\mathbb{TP}^2$.*

This manifold is called the dual projective plane and denoted with $(\mathbb{TP}^2)^*$.

PROOF. Note that from the algebraic point of view the statement is trivial. Indeed, any line is given by a polynomial "ax+by+c", $a, b, c \in \mathbb{T}$ up to the simultaneous multiplication of the coefficients a, b, c by the same

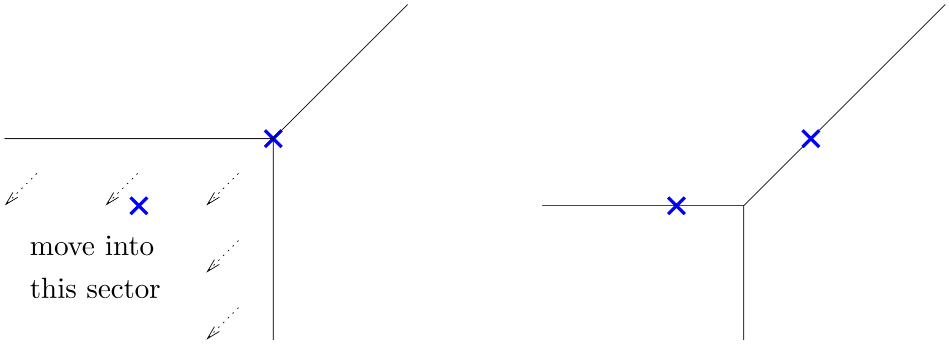

Figure 6. Finding a line passing via two points in $\mathbb{TP}^2$.

scalar $\lambda \in \mathbb{T}^\times$. These triples of coefficients up to such rescaling form $\mathbb{TP}^2$ by the very definition. Nevertheless, it is useful to look at the space of lines from a geometric point of view. A chart near a line with the 3-valent vertex in $\mathbb{R}^2$ is given by that 3-valent vertex itself.

Consider now those degenerate lines that do not coincide with a coordinate axes (those given by a binomial). These lines pass through a vertex of the triangle T and a point on its side. We still have such a distinguished point by tracing the limit of the 3-valent vertex under its approximation by non-degenerate lines, but this point is the vertex of T, so it does no longer determine the position of the line. Nevertheless, in the complement of the three points corresponding to the coordinate lines we may identify the space of all lines in $\mathbb{TP}^2$ with the space of lines together with a distinguished point (a 3-valent vertex in the case of non-degenerate line and a vertex of T otherwise). Furthermore, via this distinguished point we may identify the nondegenerate lines with the points of $\mathbb{R}^2$.

Consider the inversion $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (“\frac{1}{x}”, “\frac{1}{y}”) = (-x, -y)$. This inversion does not extend to the vertices of $\mathbb{TP}^2$, but does extend to the vertices of $\mathbb{TP}^2$ enhanced with lines passing through them. This extension gives a chart to $\mathbb{T} \times \mathbb{T}^\times$ in a neighborhood of non-coordinate lines passing via the vertex of $\mathbb{TP}^2$. Note that we may easily describe the same chart in coordinates. Finally, a coordinate line L is mapped to the opposite vertex of $\mathbb{T}$ by the inversion while choosing a nearby point in the image completely determines the line nearby to L. This gives a chart to $\mathbb{T}^2$. □

Remark 3.11. Because of the inversion σ from the proof of Theorem 3.10 it is convenient to depict the dual plane with the inverted triangle, see Figure 7. As a map $(\mathbb{TP}^2)^* \dashrightarrow \mathbb{TP}^2$ the inversion σ is only partially defined. However, replacing of the vertex of $\mathbb{TP}^2$ with all lines passing through this vertex is the tropical counterpart of the blowing up of this vertex. It allows one to define a new manifold X (that is the result of blowing up of $\mathbb{TP}^2$ in all three vertices) and everywhere defined maps $X \to \mathbb{TP}^2$ and $X \to (\mathbb{TP}^2)^*$, see Figure 7.

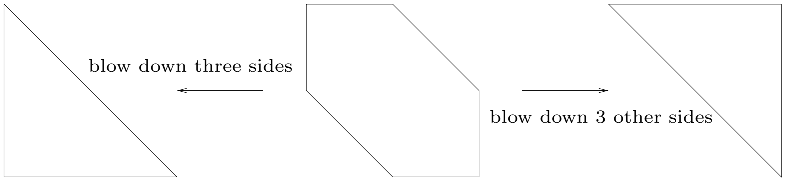

FIGURE 7. Passing from the projective plane to the dual projective plane.

Note that X is an integer affine manifold with corners as tropical blowups come with natural charts to $\mathbb{T}^2$. Furthermore, it is one of the toric varieties from Remark 1.32, the one depicted on Figure 6.

4. Curves in the plane

Let us look at the conics in $\mathbb{TP}^2$. These are the hypersurfaces given by quadratic polynomials

$$f(x,y) = \text{“}ax^2 + bxy + cy^2 + dx + ey + f\text{”},$$

$a, b, c, d, e, f \in \mathbb{T}$. We have six monomials and each can dominate the polynomial f in a certain region in the plane (possibly empty).

The Newton polygon of f is the triangle Δ_f with vertices $(0,0)$, $(2,0)$ and $(0,2)$ or its subpolygon (in the case when some of the coefficients vanish, i.e. assume the value $0_{\mathbb{T}} = -\infty$). By Theorem 3.5 there is a lattice subdivision of Δ_f for each conic $C \subset \mathbb{TP}^2$ and, conversely, each coherent subdivision of Δ_f corresponds to a conic in $\mathbb{TP}^2$.

The smallest possible convex polygon with vertices in $\mathbb{Z}^2$ is a triangle of area $\frac{1}{2}$. Such triangles are called em the primitive triangles.

DEFINITION 3.12. Curves dual to subdivision into primitive triangles are called *smooth planar tropical curves.*

Primitive triangles do not contain lattice point other than their vertices. Therefore, primitive triangulations (i.e. lattice decompositions of a Newton polygon into primitive triangles) contain all lattice points of the polygon among their vertices.

Consider a smooth conic $V_f \subset \mathbb{TP}^2$, see e.g. Figure 8. Because of the smoothness condition each monomial $m \in \Delta_f \cap \mathbb{Z}^2$ corresponds to a non-empty region in $\mathbb{TP}^2$. Furthermore, all the edges of V_f has weight 1. Let us deform just one of the coefficients of f. It is easy to see that the resulting deformation will leave the strata of V_f disjoint from m invariant. In the same time the edges of V_f corresponding to the edges of Subdiv_f adjacent

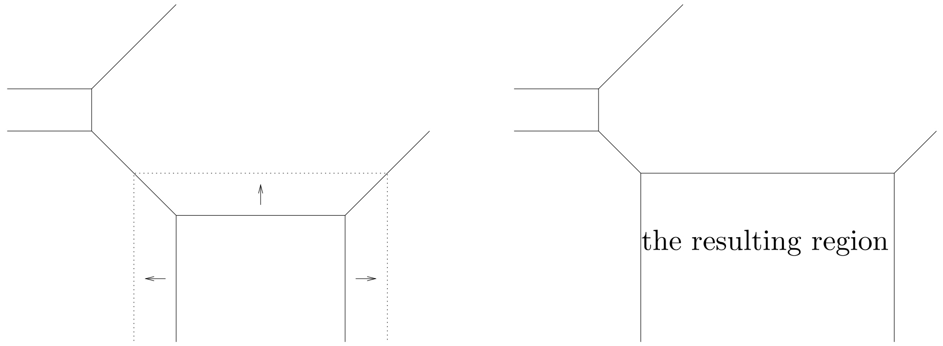

FIGURE 8. Deforming one coefficient.

to m will move enlarging or diminishing the corresponding region depending on whether we increase or decrease the coefficient of the monomial m.

Figure 10 shows some smooth conics together with the corresponding subdivisions. It is easy to see that the figure exhaust all possible combinatorial types of smooth conics.

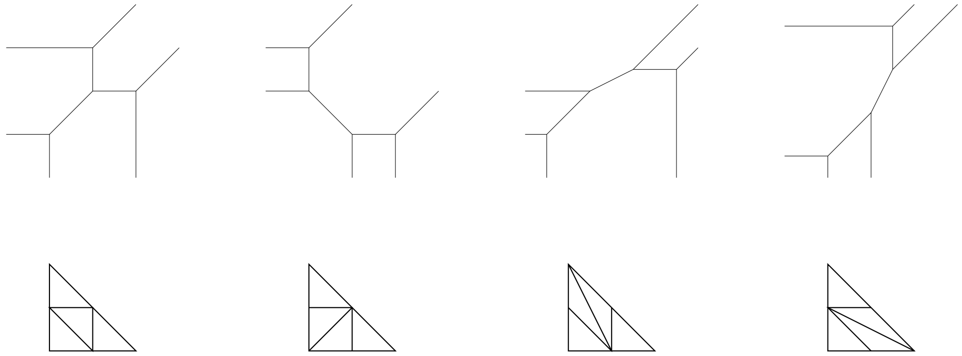

FIGURE 9. Smooth planar conics.

It is instructive to look at the possible degenerations of smooth conics. The simplest degeneration correspond to a coarser subdivision of Δ_f when we take into Subdiv_f the union of two nearby primitive triangles instead of taking each one individually. We have two combinatorially different cases: the union of two could be a parallegram or it could be a triangle of area 1, see Figure **??**. Note that the first case corresponds to a reducible conic that decomposes to the union of two lines. The second degeneration can be interpreted as a smooth conic that is tangent to a coordinate axis in $\mathbb{TP}^2$ as we shall see later.

The higher is the degree the more possibilities we have for the combinatorial type of the curve. List all combinatorial types would take too

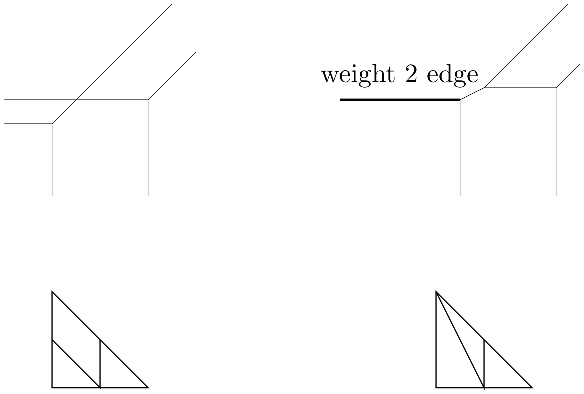

FIGURE 10. Singular planar conics.

long already for the case of planar cubic. Figure 11 depicts a smooth and a singular cubic.

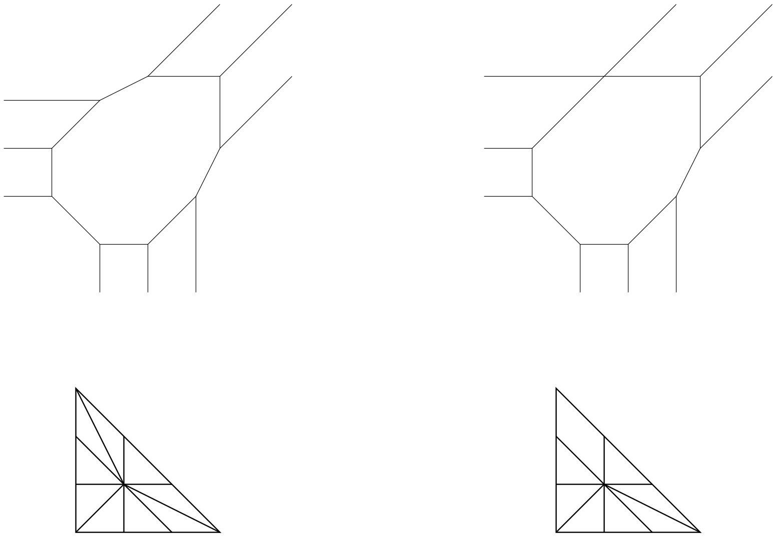

FIGURE 11. Planar cubics.

The following two examples list two particularly simple combinatorial types of smooth tropical curves of arbitrary degree. Note to specify a combinatorial type of a planar tropical curve of degree d we need to produce a lattice subdivision of the triangle $\Delta_d \subset \mathbb{R}^2$ with vertices $(0,0)$, $(d,0)$ and $(0,d)$ (or a subpolygon of this triangle).

EXAMPLE 3.13. Consider the square lattice in $\mathbb{Z}^2$. If we subdivide each square into two triangles by the diagonal parallel to the line $x+y=0$ we get a subdivision of $\mathbb{R}^2$ that is compatible with Δ_d for any d. The resulting subdivision and the tropical curve in the corresponding combinatorial type are pictured on Figure 12. The tropical curves in this combinatorial type (as well as all their degenerations) are called *honeycombs*. They proved to be useful for a range of problems related to the Horn problem, see [**32**].

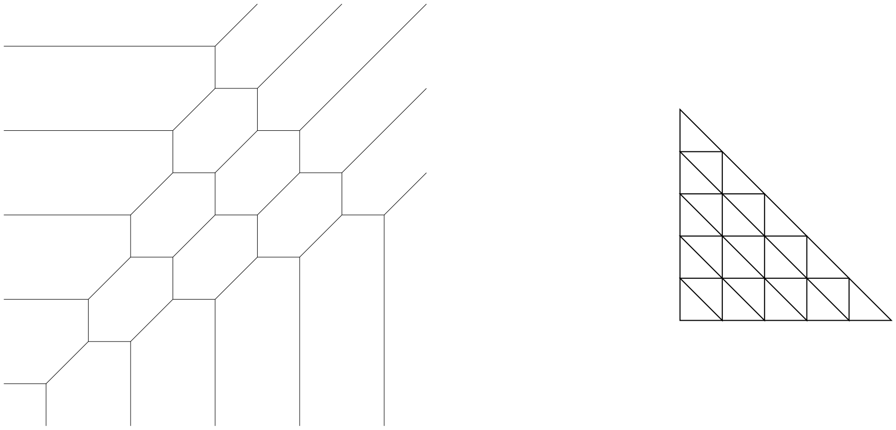

Figure 12. Honeycombs.

Note that the honeycomb triangulation of Δ_d is symmetric with respect to the exchange of the x and y coordinates. Furthermore, it is symmetric with respect to the action of the symmetric group S_3 that interchanges these two axes and the infinite axis.

Our next example is not as symmetric.

Example 3.14. Let us subdivide $\Delta_d \subset \mathbb{R}^2$ into "floors" by the lines $y = 1, \ldots, d-1$. Each floor is a trapezoid that can be further subdivided into the primitive triangles as shown on Figure 13. These subdivisions appeared in [**23**] as coherent subdivisions of higher-dimensional simplices.

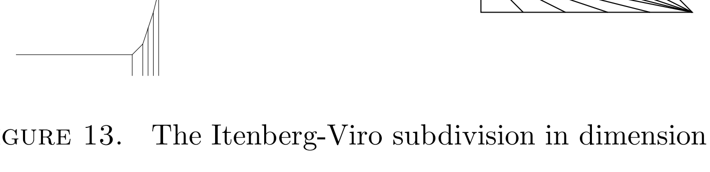

Figure 13. The Itenberg-Viro subdivision in dimension 2.

Remark 3.15. The coherence of the subdivisions in Examples 3.13 and 3.14 are verified by existence of the corresponding tropical curves. To check the latter we may note that because the lines $y = 1, \dots, d-1$ are compatible with both types of subdivision our tropical curves are glued from the curves dual to trapezoids of height 1 as shown on Figure 14. These curves are called floors. The kth floor has $d-k$ vertical rays pointing up, $d-k+1$ rays pointing down and no other vertical edges.

Furthermore, we may fix any positions (i.e. the x-coordinates) for the vertical rays pointing down and find a smooth tropical curve in the needed combinatorial type with such rays. Because of that we may inductively stack a $k+1$th floor on top of the kth floor. In particular we may combine the floors of different combinatorial types. Note also that any lattice subdivision of a the Newton polygon of a floor (i.e. a subpolygon of the strip $k-1 \le y \le k$) is coherent.

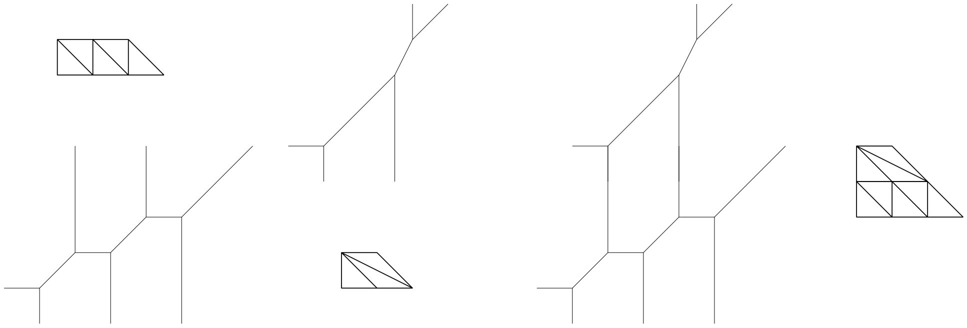

Figure 14. Floors and stacking them on top of each other.

Example 3.16. As our last example of a planar tropical curve we consider a rather involved example of a curve of degree 10. It appeared in the work of Itenberg [**21**] disproving the Ragsdale conjecture (a conjecture on topology of plane real curves that appeared in 1905 in [**55**] and was finally disproved only in 1992 [**21**]). The counterexample is provided by this very curve once we equip it with the suitable real phases, see Figure 15.

5. Surfaces in $\mathbb{TP}^3$

We start by looking at the hyperplane in $\mathbb{TP}^3$, i.e. the hypersurface given by the tropical polynomial "$ax + by + cz + d$". Similarly to the case with the lines in $\mathbb{TP}^2$ it is easy to show that any hyperplane with $a, b, c, d \neq 0_{\mathbb{T}}$ is the result of translation of the (standard) hyperplane of "$x + y + z + 1_{\mathbb{T}}$" by a vector in $\mathbb{R}^3$. Again if some (but not all) of the coefficients a, b, c, d assume the value $0_{\mathbb{T}}$ then we can interpret the corresponding hyperplane as the limiting set of a family of translations of $V_{\text{“}x+y+z+1_{\mathbb{T}}\text{”}}$ in $\mathbb{R}^3$.

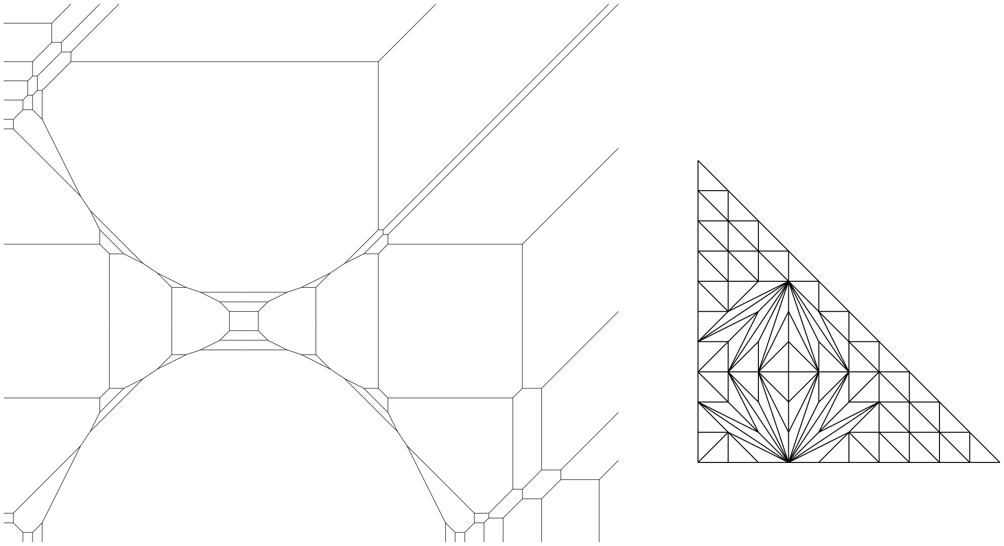

FIGURE 15. The Itenberg-Ragsdale curve of degree 10.

Figure 16 depicts a generic hyperplane $H \subset \mathbb{TP}^2$. It consists of 6 sectors, all of them have a common vertex $v \in \mathbb{R}^3$. There are 4 outgoing rays from v, in the direction $(-1,0,0)$, $(0,-1,0)$, $(0,0,-1)$ and $(1,1,1)$. Any pair of these rays span a sector in $\mathbb{R}^3$ diffeomorphic to the positive quadrant $\mathbb{R}^2_{\geq 0}$. To get H we take the closure in $\mathbb{TP}^3 \supset \mathbb{R}^3$ of the union of the 6 sectors.

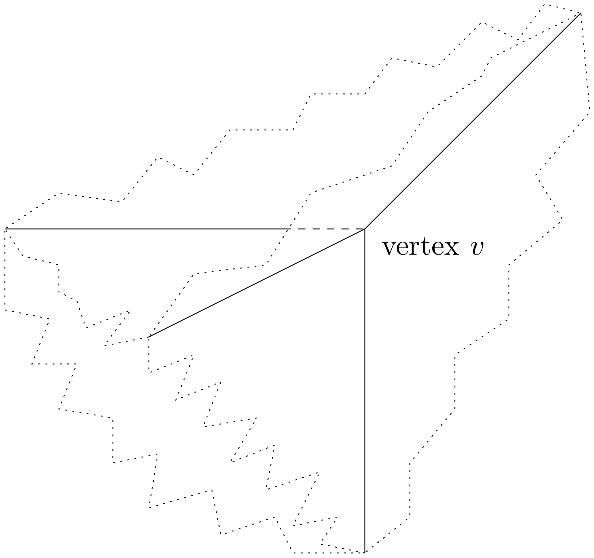

FIGURE 16. A tropical plane in the 3-space.

The position of the vertex $v \in \mathbb{R}^3$ completely determines a tropical hyperplane $V_{\text{“}ax+by+cz+d\text{”}}$ with $a, b, c, d \neq 0_{\mathbb{T}}$. Similarly to the case with lines in $\mathbb{TP}^2$ all hyperplanes are parameterized by the dual space $(\mathbb{TP}^3)^* \supset \mathbb{R}^3$. More generally we have the following statement generalizing Theorem 3.10.

Theorem 3.17. *The space of all hyperplanes (hypersurfaces given by tropical polynomials of degree 1) in $\mathbb{TP}^n$ forms an integer affine manifold with corners isomorphic to $\mathbb{TP}^n$.*

Proof. Let us first note that the Theorem is trivial if $n = 1$. Indeed a hyperplane in $\mathbb{TP}^1$ is given by a polynomial “$ax + b$” $= \max a + x, b$ in one variable x, $a, b \in \mathbb{T}$, “ab” $\neq 0_{\mathbb{T}}$. The corresponding hypersurface always just the single point $x = b - a \in \mathbb{TP}^1$. Thus the set of such hyperplanes coincides with the set of points in $\mathbb{TP}^1$.

To prove the theorem in general it suffices to prove show that if f is a polynomial of degree 1 in n variables then the set V_f determines the coefficients of f up to their simultaneous tropical multiplication by a non-zero constant. Indeed, once we prove this we can identify the space of hyperplanes with the space of all coefficients up to the simultaneous rescaling which is the tropical projective n-space by definition.

Recall that $\mathbb{TP}^n$ is topologically a simplex. Each edge of this simplex corresponds to a tropical line $\mathbb{TP}^1$ obtained as the intersection of $(n-1)$ coordinate planes. The hyperplane V_f cuts a point on each such $\mathbb{TP}^1$ unless this $\mathbb{TP}^1$ is contained in V_f. Each such point is a hyperplane in $\mathbb{TP}^1$ and determines two coefficients of f up to scaling. If the line $\mathbb{TP}^1$ is contained in V_f then both corresponding coefficients must be equal to $0_{\mathbb{T}}$. $\square$

We call this space of hyperplanes the dual projective space and denote with $(\mathbb{TP}^n)^*$.

To understand the geometry of higher-degree surfaces in $\mathbb{TP}^3$ (and more generally the geometry of higher-dimensional tropical varieties) it is convenient to introduce the notion of *floor decomposition.*

...

(TO BE CONTINUED)

...

6. Complete Intersections

7. Balancing condition

CHAPTER 4

Tropical varieties

CHAPTER 5

Tropical equivalence

Bibliography

[1] O. Aharony, A. Hanany, *Branes, superpotentials and superconformal fixed points*, http://arxiv.org hep-th/9704170.

[2] O. Aharony, A. Hanany, B. Kol, *Webs of (p, q) 5-branes, five dimensional field theories and grid diagrams*, http://arxiv.org hep-th/9710116.

[3] V. I. Arnold, *The situation of ovals of real plane algebraic curves, the involutions of four-dimensional smooth manifolds, and the arithmetic of integral quadratic forms*, Funkcional. Anal. i Priložen. **5** (1971) no. 3, 1-9.

[4] M. F. Atiyah, *Angular momentum, convex polyhedra and algebraic geometry*, Proc. Edinburgh Math. Soc., **26** (1983) 121-133.

[5] J. P. Benzecri, *Variétés localment affines*, Séminaire Ehresmann, May 1959.

[6] G. M. Bergman, *The logarithmic limit set of an algebraic variety*, Trans. AMS, **157** (1971), 459-469.

[7] D. Bernstein, *The number of roots of a system of equations*, Functional Anal. Appl. **9** (1975), 183-185.

[8] T. Bogart, A. Jensen, D. Speyer, B. Sturmfels, R. Thomas, *Computing Tropical Varieties*, http://arxiv.org/abs/math.AG/0507563.

[9] F. Bourgeois, *A Morse-Bott approach to contact homology*, Dissertation, Stanford University 2002.

[10] L. Brusotti, *Curve generatrici e curve aggregate nella costruzione di curve piane d'ordine assegnato dotate del massimo numero di circuiti,* Rend. Circ. Mat. Palermo **42** (1917), 138-144.

[11] M. Einsiedler, M. Kapranov, D. Lind, *Non-archimedean amoebas and tropical varieties*, http://arxiv.org math.AG/0408311.

[12] Y. Eliashberg, A. Givental, H. Hofer, *Introduction to symplectic field theory,* GAFA 2000, Special Volume, Part II, 560–673.

[13] M. Forsberg, M. Passare, A. Tsikh, *Laurent determinants and arangements of hyperplane amoebas,* Advances in Math. **151** (2000), 45–70.

[14] K. Fukaya, *Multivalued Morse theory, asymptotic analysis and mirror symmetry,* Graphs and patterns in mathematics and theoretical physics, 205–278, Proc. Sympos. Pure Math., **73**, Amer. Math. Soc., Providence, RI, 2005.

[15] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants.* Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1994.

[16] , M. Gross, B. Siebert, *Affine structures...*

[17] M. Gross, P. M. H. Wilson, *Large complex structure limits of $K3$ surfaces,* J. Differential Geom. **55** (2000), no. 3, 475–546.

[18] A. Harnack, *Über Vieltheiligkeit der ebenen algebraischen Curven,* Math. Ann. **10** (1876), 189-199.

[19] A. Henriques, *An analogue of convexity for complements of amoebas of varieties of higher codimensions*, Preprint, Berkeley, May 2001.

[20] D. Hilbert, *Mathematische Probleme*, Arch. Math. Phys. (3) **1** (1901), 213-237.

[21] I. Itenberg,...

[22] I. Itenberg, V. Kharlamov, E. Shustin, *Welschinger invariant and enumeration of real plane rational curves,* http://arxiv.org/abs/math.AG/0303378.

[23] I. Itenberg, O. Viro, *Patchworking real algebraic curves disproves the Ragsdale conjecture*, Math. Intelligencer, **18** (1996), 19-28.

[24] I. Itenberg, O. Viro, *Maximal real algebraic hypersurfaces of projective space*, to appear.

[25] M. M. Kapranov, *A characterization of A-discriminantal hypersurfaces in terms of the logarithmic Gauss map*, Math. Ann. **290** (1991), 277-285

[26] M. M. Kapranov, *Amoebas over non-Archimedean fields*, Preprint, 2000.

[27] R. Kenyon, A. Okounkov, *Planar dimers and Harnack curves*, http://arxiv.org math.AG/0311062.

[28] R. Kenyon, A. Okounkov, *Limit shapes and the complex burgers equation,* http://arxiv.org math-ph/0507007.

[29] R. Kenyon, A. Okounkov, S. Sheffield, *Dimers and amoebae*, http://arxiv.org math-ph/0311005.

[30] V. M. Kharlamov, *On the classification of nonsingular surfaces of degree 4 in* $\mathbb{RP}^3$ *with respect to rigid isotopies*, Funktsional. Anal. i Prilozhen. **18** (1984), no. 1, 49-56.

[31] A. G. Khovanskii, *Newton polyhedra and toric varieties,* Funkcional. Anal. i Priložen. **11** (1977), no. 4, 56 - 64.

[32] , A. Knutson, T. Tao,...

[33] M. Kontsevich, Yu. Manin, *Gromov-Witten classes, quantum cohomology and enumerative geometry*, Comm. Math. Phys. **164** (1994), 525-562.

[34] M. Kontsevich, Y. Soibelman, *Homological mirror symmetry and torus fibrations,* Symplectic geometry and mirror symmetry (Seoul, 2000), 203–263, World Sci. Publishing, River Edge, NJ, 2001.

[35] M. Kontsevich, Y. Soibelman, *Affine structures and non-archimedean analytic spaces,* http://arxiv.org math.AG/0406564.

[36] N. H. Kuiper, *Sur les surfaces localement affines*, Géométrie Différentielle, Colloq. ICNRS, CNRS, Paris 1953, 79–87.

[37] V. P. Maslov, *New superposition principle for optimization problems*, in Seminaire sur les Equations auc Dérivées Partielles 1985/6, Centre Mathématiques de l'École Polytechnique, Palaiseau, 1986, exposé 24.

[38] G. Mikhalkin, *Real algebraic curves, moment map and amoebas*, Ann. of Math. **151** (2000), 309 - 326.

[39] G. Mikhalkin, H. Rullgård, *Amoebas of maximal area*, Intern. Math. Res. Notices **9** (2001), 441-451.

[40] G. Mikhalkin, *Amoebas of algebraic varieties,* a report for the Real Algebraic and Analytic Geometry congress, June 2001, Rennes, France, http://arxiv.org math.AG/0108225.

[41] G. Mikhalkin, *Decomposition into pairs-of-pants for complex algebraic hypersurfaces*, http://arxiv.org math.GT/0205011 Preprint 2002, to appear in Topology.

[42] G. Mikhalkin, *Counting curves via lattice paths in polygons*, C. R. Math. Acad. Sci. Paris **336** (2003), no. 8, 629-634.

[43] G. Mikhalkin, *Enumerative tropical geometry in* $\mathbb{R}^2$*,* J. Amer. Math. Soc. **18** (2005), no. 2, 313–377.

[44] G. Mikhalkin, *Maximal real algebraic hypersurfaces* (in preparation).

[45] G. Mikhalkin, A. Okounkov, *Geometry of shock curves in toric surfaces* (in preparation).

[46] G. Mikhalkin, I. Zharkov, *Tropical curves, their Jacobicans and* Θ*-functions* (in preparation).

[47] T. Nagano, K. Yagi, *The affine structures on the real two-torus I*, Bull. AMS, **79**, no. 6 (1973), 1251–1253.

[48] V. Nikulin, I. Shafarevich, book

[49] T. Nishinou, B. Siebert, *Toric degenerations of toric varieties and tropical curves,* http://arxiv.org math.AG/0409060.

[50] L. Pachter, B. Sturmfels, *Tropical geometry of statistical models*, Proc. Natl. Acad. Sci. USA **101** (2004), no. 46, 16132–16137.

[51] M. Passare, H. Rullgård, *Amoebas, Monge-Ampère measures, and triangulations of the Newton polytope*. Preprint, Stockholm University, 2000.

[52] M. Passsare, A. Tsikh, *Passare, Mikael; Tsikh, August Amoebas: their spines and their contours,* Idempotent mathematics and mathematical physics, 275–288, Contemp. Math., **377**, Amer. Math. Soc., Providence, RI, 2005.

[53] I. G. Petrovsky, *On the topology of real plane algebraic curves*, Ann. Math. **39** (1938), 187-209.

[54] J.-E. Pin, *Tropical semirings,* Idempotency (Bristol, 1994), 50–69, Publ. Newton Inst., 11, Cambridge Univ. Press, Cambridge, 1998.

[55] Ragsdale,...

[56] J. Richter-Gebert, B. Sturmfels, Th. Theobald, *First steps in tropical geometry*, Idempotent mathematics and mathematical physics, 289–317, Contemp. Math., **377**, Amer. Math. Soc., Providence, RI, 2005.

[57] V. A. Rohlin, *Congruences modulo* 16 *in Hilbert's sixteenth problem*, Funkcional. Anal. i Priložen. **6** (1972), no. 4, 58-64.

[58] L. Ronkin, *On zeroes of almost periodic functions generated by holomorphic functions in a multicircular domain*, to appear in "Complex Analysis in Modern Mathematics", Fazis, Moscow, 2000, 243-256.

[59] H. Rullgård, *Stratification des espaces de polynômes de Laurent et la structure de leurs amibes*, C. R. Acad. Sci. Paris, Série I, **331** (2000), 355-358.

[60] H. Rullgård, *Polynomial amoebas and convexity*, Preprint, Stockholm University, 2001.

[61] E. Shustin, *A tropical calculation of the Welschinger invariants of real toric Del Pezzo surfaces*, http://arxiv.org math.AG/0406099.

[62] E. Shustin, Z. Izhakian, *A tropical Nullstellensatz*, http://arxiv.org math.AC/0508413.

[63] D. Speyer, *Horn's problem, Vinnikov curves, and the hive cone,* Duke Math. J. **127** (2005), no. 3, 395–427.

[64] D. Speyer, B. Sturmfels, *The tropical Grassmanian*, Adv. Geom. **4** (2004), no. 3, 389–411.

[65] D. Speyer, B. Sturmfels, *Tropical Mathematics,* http://arxiv.org math.CO/0408099.

[66] B. Sturmfels, *Solving systems of polynomial equations*, CBMS Regional Conference Series in Mathematics, **97** Providence, RI, 2002.

[67] W. Thurston, book.

[68] O. Ya. Viro, *Patchworking*, 1979.

[69] O. Ya. Viro, *Real plane algebraic curves: constructions with controlled topology,* Leningrad Math. J. **1** (1990), no. 5, 1059-1134.

[70] O. Ya. Viro, *Dequantization of Real Algebraic Geometry on a Logarithmic Paper,* Proceedings of the European Congress of Mathematicians (2000).

[71] G. Wilson, *Hilbert's sixteenth problem*, Topology **17** (1978), 53-73.