2.1 Tangent morphism

The tangent bundle itself is only the result of applying the tangent functor to a manifold. We must explain how to apply the tangent functor to a morphism of manifolds. This is otherwise known as taking the “derivative” of a smooth map \( f : M \to N \). Such a map may be defined locally in charts \((U_i, \varphi_i)\) for \(M\) and \((V_i, \psi_i)\) for \(N\) as a collection of vector-valued functions \( \psi_i \circ f \circ \varphi_i^{-1} = f_{i\alpha} : \varphi_i(U_i) \to \psi_i(V_\alpha) \) which satisfy

\[
\psi_{i\beta} \circ f_{i\alpha} = f_{i\beta} \circ \psi_{j\gamma}.
\]

Differentiating, we obtain

\[
D\psi_{i\beta} \circ Df_{i\alpha} = Df_{i\beta} \circ D\psi_{j\gamma},
\]

and hence we obtain a map \( TM \to TN \). This map is called the derivative of \( f \) and is denoted \( Tf : TM \to TN \) (or sometimes just \( Df : TM \to TN \)). Sometimes it is called the “push-forward” of vectors and is denoted \( f_* \).

The map fits into the commutative diagram

\[
\begin{array}{ccc}
TM & \xrightarrow{Tf} & TN \\
\pi_M & \downarrow & \downarrow \pi_N \\
M & \xrightarrow{f} & N
\end{array}
\]

Just as \( \pi^{-1}(x) = T_x M \subseteq TM \) is a vector space for all \( x \), making \( TM \) into a “bundle of vector spaces”, the map \( Tf : T_x M \to T_{f(x)} N \) is a linear map and hence \( Tf \) is a “bundle of linear maps”. The pair \((f, Tf)\) is a morphism of vector bundles \((TM, \pi_M, M) \to (TN, \pi_N, N)\).

The usual chain rule for derivatives then implies that if \( f \circ g = h \) as maps of manifolds, then \( Tf \circ Tg = Th \).

As a result, we obtain the following category-theoretic statement.

**Proposition 2.2.** The map \( T \) which takes a manifold \( M \) to its tangent bundle \( TM \), and which takes maps \( f : M \to N \) to the derivative \( Tf : TM \to TN \), is a functor from the category of manifolds and smooth maps to itself.

The tangent bundle allows us to make sense of the notion of vector field in a global way. Locally, in a chart \((U_i, \varphi_i)\), we would say that a vector field \( X_i \) is simply a vector-valued function on \( U_i \), i.e. a function \( X_i : \varphi(U_i) \to \mathbb{R}^n \). Of course if we had another vector field \( X_j \) on \((U_j, \varphi_j)\), then the two would agree as vector fields on the overlap \( U_i \cap U_j \) when \( D\varphi_{ij} : X_i(p) \to X_j(\varphi_j(p)) \). So, if we specify a collection \( \{X_i \in C^\infty(U_i, \mathbb{R}^n)\} \) which glue on overlaps, this would define a global vector field. This leads precisely to the following definition.

**Definition 12.** A smooth vector field on the manifold \( M \) is a smooth map \( X : M \to TM \) such that \( \pi \circ X : M \to M \) is the identity. Essentially it is a smooth assignment of a unique tangent vector to each point in \( M \).

Such maps \( X \) are also called cross-sections or simply sections of the tangent bundle \( TM \), and the set of all such sections is denoted \( C^\infty(M, TM) \) or sometimes \( \Gamma^\infty(M, TM) \), to distinguish them from simply smooth maps \( M \to TM \).

**Example 2.3.** From a computational point of view, given an atlas \((\tilde{U}_i, \varphi_i)\) for \( M \), let \( U_i = \varphi_i(\tilde{U}_i) \subseteq \mathbb{R}^n \) and let \( \varphi_{ij} = \varphi_j \circ \varphi_{i}^{-1} \). Then a global vector field \( X \in \Gamma^\infty(M, TM) \) is specified by a collection of vector-valued functions \( X_i : U_i \to \mathbb{R}^n \) such that \( D\varphi_{ij}(X_i(x)) = X_j(\varphi_{ij}(x)) \) for all \( x \in \varphi_i(U_i \cap U_j) \).

For example, if \( S^1 = U_0 \cap U_1 / \sim \), with \( U_0 = \mathbb{R} \) and \( U_1 = \mathbb{R} \), with \( x \in U_0 \setminus \{0\} \sim y \in U_1 \setminus \{0\} \) whenever \( y = x^{-1} \), then \( \varphi_{01} : x \mapsto x^{-1} \) and \( D\varphi_{01}(x) : (x, v) \mapsto (x^{-1}, -x^{-2} v) \).

If we choose the coordinate vector field \( X_0 = \frac{\partial}{\partial x} \) (in coordinates this is simply \( x \mapsto (x, 1) \)), then we see that \( D\varphi_{01}(X_0) = (x^{-1}, -x^{-2} \cdot 1) \), i.e. the vector field \( y \mapsto (y, -y^2) \), in other words \( X_1 = -y^2 \frac{\partial}{\partial y} \).

Hence the following local vector fields glue to form a global vector field on \( S^1 \):

\[
X_0 = \frac{\partial}{\partial x}, \\
X_1 = -y^2 \frac{\partial}{\partial y}.
\]
This vector field does not vanish in \( U_0 \) but vanishes to order 2 at a single point in \( U_1 \). Find the local expression in these charts for the rotational vector field on \( S^1 \) given in polar coordinates by \( \frac{\partial}{\partial \theta} \).

A word of warning: it may be tempting to think that the assignment \( M \mapsto \Gamma^\infty(M, TM) \) is a functor from manifolds to vector spaces; it is not, because there is no way to push forward or pull back vector fields. Nevertheless, if \( f : M \rightarrow N \) is a smooth map, it does define an equivalence relation between vector fields on \( M \) and \( N \):

**Definition 13.** if \( f : M \rightarrow N \) smooth, then \( X \in \Gamma^\infty(M, TM) \) is called \( f \)-related to \( Y \in \Gamma^\infty(N, TN) \) when \( f_* (X(p)) = Y(f(p)) \), i.e. the diagram commutes:

\[
\begin{array}{ccc}
TM & \xrightarrow{Tf} & TN \\
\downarrow x & & \downarrow y \\
M & \xrightarrow{f} & N
\end{array}
\]

So, another way to phrase the definition of a vector field is that they are local vector-valued functions which are \( \varphi_i \)-related on overlaps.

### 2.2 Properties of vector fields

The concept of derivation of an algebra \( A \) is the infinitesimal version of an automorphism of \( A \). That is, if \( \phi_t : A \rightarrow A \) is a family of automorphisms of \( A \) starting at \( \text{Id} \), so that \( \phi_t(ab) = \phi_t(a)\phi_t(b) \), then the map \( a \mapsto \frac{d}{dt}|_{t=0} \phi_t(a) \) is a derivation.

**Definition 14.** A derivation of the \( \mathbb{R} \)-algebra \( A \) is a \( \mathbb{R} \)-linear map \( D : A \rightarrow A \) such that \( D(ab) = (Da)b + a(Db) \). The space of all derivations is denoted \( \text{Der}(A) \). Note that this makes sense for noncommutative algebras also.

In the following, we show that derivations of the algebra of functions actually correspond to vector fields.

The vector fields \( \Gamma^\infty(M, TM) \) form a vector space over \( \mathbb{R} \) of infinite dimension (unless \( \dim M = 0 \)). They also form a module over the ring of smooth functions \( \Gamma^\infty(M, \mathbb{R}) \) via pointwise multiplication: for \( f \in \Gamma^\infty(M, \mathbb{R}) \) and \( X \in \Gamma^\infty(M, TM) \), we claim that \( fX = x \mapsto f(x)X(x) \) defines a smooth vector field: this is clear from local considerations: A vector field \( V = \sum i v^i (x^1, \ldots, x^n) \frac{\partial}{\partial x^i} \) is smooth precisely when the component functions \( v^i \) are smooth: the vector field \( TV \) then has components \( f v^i \), still smooth.

The important property of vector fields which we are interested in is that they act as \( \mathbb{R} \)-derivations of the algebra of smooth functions. Locally, it is clear that a vector field \( X = \sum_i a^i \frac{\partial}{\partial x^i} \) gives a derivation of the algebra of smooth functions, via the formula \( X(f) = \sum_i a^i \frac{\partial f}{\partial x^i} \), since

\[
X(fg) = \sum_i a^i \left( \frac{\partial f}{\partial x^i} g + f \frac{\partial g}{\partial x^i} \right) = X(f)g + fX(g).
\]

We wish to verify that this local action extends to a well-defined global derivation on \( \Gamma^\infty(M, \mathbb{R}) \).

**Proposition 2.4.** Let \( f \) be a smooth function on \( U \subset \mathbb{R}^n \), and \( X : U \rightarrow T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \) a vector field. Then

\[
X(f) = \pi_2 \circ Df \circ X,
\]

where \( \pi_2 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is the second projection (i.e. projection to the fiber of \( T\mathbb{R}^n \)). In local coordinates, we have \( X(f) = \sum_i a^i \frac{\partial}{\partial x^i} \) whereas \( Df : X(x) \mapsto (f(x), \sum_i \frac{\partial f}{\partial x^i} a^i) \), so that we obtain the result by projection.

**Proposition 2.5.** Local partial differentiation extends to an injective map \( \Gamma^\infty(M, TM) \rightarrow \text{Der}(\Gamma^\infty(M, \mathbb{R})) \).
Proof. A global function is given by \( f_i = f_j \circ \varphi_{ij} \). We verify that

\[
\begin{align*}
X_i(f_i) &= \pi_2 \circ D f_i \circ X_i \\
&= \pi_2 \circ D f_j \circ D \varphi_{ij} \circ X_i \\
&= \pi_2 \circ D f_j \circ X_j \circ \varphi_{ij} \\
&= X_j(f_j) \circ \varphi_{ij},
\end{align*}
\]

showing that \( \{X_i(f_i)\} \) defines a global function. Injectivity follows from the local fact that \( V(f) = 0 \) for all \( f \) would imply, for \( V = \sum_i v^i \frac{\partial}{\partial x^i} \), that \( V(x^i) = v^i = 0 \) for all \( i \), i.e. \( V = 0 \).

\( \square \)