1 Manifolds

A manifold is a space which looks like $\mathbb{R}^n$ at small scales (i.e. "locally"), but which may be very different from this at large scales (i.e. "globally"). In other words, manifolds are made up by gluing pieces of $\mathbb{R}^n$ together to make a more complicated whole. We would like to make this precise.

1.1 Topological manifolds

Definition 1. A real, n-dimensional *topological manifold* is a Hausdorff, second countable topological space which is locally homeomorphic to $\mathbb{R}^n$.

Note: "Locally homeomorphic to $\mathbb{R}^n$" simply means that each point p has an open neighbourhood U for which we can find a homeomorphism $\varphi : U \longrightarrow V$ to an open subset $V \in \mathbb{R}^n$. Such a homeomorphism φ is called a *coordinate chart* around p. A collection of charts which cover the manifold, i.e. whose union is the whole space, is called an *atlas*.

We now give a bunch of examples of topological manifolds. The simplest is, technically, the empty set. More simple examples include a countable set of points (with the discrete topology), and $\mathbb{R}^n$ itself, but there are more:

Example 1.1 (Circle)**.** *Define the circle $S^1 = \{z \in \mathbb{C} \ : \ |z| = 1\}$. Then for any fixed point $z \in S^1$, write it as $z = e^{2\pi i c}$ for a unique real number $0 \leq c < 1$, and define the map*

$$\nu_z : t \mapsto e^{2\pi i t}. \tag{1}$$

We note that ν_z maps the interval $I_c = (c - \frac{1}{2}, c + \frac{1}{2})$ to the neighbourhood of z given by $S^1 \backslash \{-z\}$, and it is a homeomorphism. Then $\varphi_z = \nu_z|_{I_c}^{-1}$ is a local coordinate chart near z.

By taking products of coordinate charts, we obtain charts for the Cartesian product of manifolds. Hence the Cartesian product is a manifold.

Example 1.2 (n-torus)**.** *$S^1 \times \cdots \times S^1$ is a topological manifold (of dimension given by the number n of factors), with charts $\{\varphi_{z_1} \times \cdots \times \varphi_{z_n} \ : \ z_i \in S^1\}$.*

Example 1.3 (open subsets)**.** *Any open subset $U \subset M$ of a topological manifold is also a topological manifold, where the charts are simply restrictions $\varphi|_U$ of charts φ for M.*

For example, the real $n \times n$ matrices $\mathrm{Mat}(n, \mathbb{R})$ form a vector space isomorphic to $\mathbb{R}^{n^2}$, and contain an open subset

$$GL(n, \mathbb{R}) = \{A \in \mathrm{Mat}(n, \mathbb{R}) \ : \ \det A \neq 0\}, \tag{2}$$

known as the general linear group, which therefore forms a topological manifold.

Example 1.4 (Spheres)**.** *The n-sphere is defined as the subspace of unit vectors in $\mathbb{R}^{n+1}$:*

$$S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \ : \ \sum x_i^2 = 1\}.$$

Let $N = (1, 0, \ldots, 0)$ be the North pole and let $S = (-1, 0, \ldots, 0)$ be the South pole in S^n. Then we may write S^n as the union $S^n = U_N \cup U_S$, where $U_N = S^n \backslash \{S\}$ and $U_S = S^n \backslash \{N\}$ are equipped with coordinate charts φ_N, φ_S into $\mathbb{R}^n$, given by the "stereographic projections" from the points S, N respectively

$$\varphi_N : (x_0, \vec{x}) \mapsto (1 + x_0)^{-1}\vec{x}, \tag{3}$$

$$\varphi_S : (x_0, \vec{x}) \mapsto (1 - x_0)^{-1}\vec{x}. \tag{4}$$

We have endowed the sphere S^n with a certain topology, but is it possible for another topological manifold $\tilde{S}^n$ to be homotopic to S^n without *being homeomorphic to it? The answer is no, and this is known as the topological*

Poincaré conjecture, and is usually stated as follows: any homotopy n-sphere is homeomorphic to the n-sphere. It was proven for $n > 4$ by Smale, for $n = 4$ by Freedman, and for $n = 3$ is equivalent to the smooth Poincaré conjecture which was proved by Hamilton-Perelman. In dimensions $n = 1, 2$ it is a consequence of the (easy) classification of topological 1- and 2-manifolds.

Example 1.5 (Projective spaces)**.** *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then $\mathbb{K}P^n$ is defined to be the space of lines through $\{0\}$ in $\mathbb{K}^{n+1}$, and is called the projective space over $\mathbb{K}$ of dimension n.*

More precisely, let $X = \mathbb{K}^{n+1}\backslash\{0\}$ and define an equivalence relation on X via $x \sim y$ iff $\exists\lambda \in \mathbb{K}^* = \mathbb{K}\backslash\{0\}$ such that $\lambda x = y$, i.e. x, y lie on the same line through the origin. Then

$$\mathbb{K}P^n = X/\sim,$$

and it is equipped with the quotient topology.

The projection map $\pi : X \longrightarrow \mathbb{K}P^n$ is an open *map, since if $U \subset X$ is open, then tU is also open $\forall t \in \mathbb{K}^*$, implying that $\cup_{t\in\mathbb{K}^*} tU = \pi^{-1}(\pi(U))$ is open, implying $\pi(U)$ is open. This immediately shows, by the way, that $\mathbb{K}P^n$ is second countable.*

To show $\mathbb{K}P^n$ is Hausdorff (which we must do, since Hausdorff is preserved by subspaces and products, but not *quotients), we would like to show that the diagonal in $\mathbb{K}P^n \times \mathbb{K}P^n$ is closed. We show this by showing that the graph of the equivalence relation is closed in $X \times X$ (this, together with the openness of π, gives us the result). This graph is simply*

$$\Gamma_\sim = \{(x, y) \in X \times X \ : \ x \sim y\},$$

and we notice that $\Gamma_\sim$ is actually the common zero set of the following continuous functions

$$f_{ij}(x, y) = (x_i y_j - x_j y_i) \quad i \neq j.$$

An atlas for $\mathbb{K}P^n$ is given by the open sets $U_i = \pi(\tilde{U}_i)$, where

$$\tilde{U}_i = \{(x_0, \dots, x_n) \in X \ : \ x_i \neq 0\},$$

and these are equipped with charts to $\mathbb{K}^n$ given by

$$\varphi_i([x_0, \dots, x_n]) = x_i^{-1}(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \tag{5}$$

which are indeed invertible by $(y_1, \dots, y_n) \mapsto (y_1, \dots, y_i, 1, y_{i+1}, \dots, y_n)$.

Sometimes one finds it useful to simply use the "coordinates" $(x_0, \dots, x_n)$ for $\mathbb{K}P^n$, with the understanding that the x_i are well-defined only up to overall rescaling. This is called using "projective coordinates" and in this case a point in $\mathbb{K}P^n$ is denoted by $[x_0 : \cdots : x_n]$.

Example 1.6 (Connected sum)**.** *Let $p \in M$ and $q \in N$ be points in topological manifolds and let (U, φ) and (V, ψ) be charts around p, q such that $\varphi(p) = 0$ and $\psi(q) = 0$.*

Choose ϵ small enough so that $B(0, 2\epsilon) \subset \varphi(U)$ and $B(0, 2\epsilon) \subset \varphi(V)$, and define the map of annuli

$$\phi : B(0, 2\epsilon)\backslash\overline{B(0, \epsilon)} \longrightarrow B(0, 2\epsilon)\backslash\overline{B(0, \epsilon)} \tag{6}$$

$$x \mapsto \tfrac{2\epsilon^2}{|x|^2} x. \tag{7}$$

This is a homeomorphism of the annulus to itself, exchanging the boundaries. Now we define a new topological manifold, called the connected sum *$M\sharp N$, as the quotient $X/\sim$, where*

$$X = (M\backslash\overline{\varphi^{-1}(B(0, \epsilon))}) \sqcup (N\backslash\overline{\psi^{-1}(B(0, \epsilon))}),$$

and we define an identification $x \sim \psi^{-1}\phi\varphi(x)$ for $x \in \varphi^{-1}(B(0, 2\epsilon))$. If $\mathcal{A}_M$ and $\mathcal{A}_N$ are atlases for M, N respectively, then a new atlas for the connect sum is simply

$$\mathcal{A}_M|_{M\backslash\overline{\varphi^{-1}(B(0,\epsilon))}} \cup \mathcal{A}_N|_{N\backslash\overline{\psi^{-1}(B(0,\epsilon))}}$$

Two important remarks concerning the connect sum: first, the connect sum of a sphere with itself is homeomorphic to the same sphere:

$$S^n \sharp S^n \cong S^n.$$

Second, by taking repeated connect sums of T^2 and $\mathbb{R}P^2$, we may obtain all compact 2-dimensional manifolds.

Example 1.7. *Let F be a topological space. A fiber bundle with fiber F is a triple (E, p, B), where E, B are topological spaces called the "total space" and "base", respectively, and $p : E \longrightarrow B$ is a continuous surjective map called the "projection map", such that, for each point $b \in B$, there is a neighbourhood U of b and a homeomorphism*

$$\Phi : p^{-1}U \longrightarrow U \times F,$$

such that $p_U \circ \Phi = p$, where $p_U : U \times F \longrightarrow U$ is the usual projection. The submanifold $p^{-1}(b) \cong F$ is called the "fiber over b".

When B, F are topological manifolds, then clearly E becomes one as well. We will often encounter such manifolds.

1.2 Smooth manifolds

Given coordinate charts (U_i, φ_i) and (U_j, φ_j) on a topological manifold, if we compare coordinates on the intersection $U_{ij} = U_i \cap U_j$, we see that the map

$$\varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij})$$

is a homeomorphism, simply because it is a composition of homeomorphisms. We can say this another way: topological manifolds are glued together by homeomorphisms.

This means that we may be able to differentiate a function in one coordinate chart but not in another, i.e. there is no way to make sense of calculus on topological manifolds. This is why we introduce smooth manifolds, which is simply a topological manifold where the gluing maps are required to be *smooth*.

First we recall the notion of a smooth map of finite-dimensional vector spaces.

Remark 1 (Aside on smooth maps of vector spaces)**.** *Let $U \subset V$ be an open set in a finite-dimensional vector space, and let $f : U \longrightarrow W$ be a function with values in another vector space W. The function f is said to be differentiable at $p \in U$ if there exists a linear map $Df(p) : V \longrightarrow W$ such that*

$$\lim_{||x|| \to 0} \frac{||f(p + x) - f(p) - Df(p)(x)||}{||x||} = 0.$$

Here we choose any norm[1] $||\cdot||$ *on U, V since such norms are all equivalent for finite-dimensional vector spaces. For infinite-dimensional vector spaces, the topology is highly sensitive to which norm is chosen, but we will work in finite dimensions.*

Given linear coordinates $(x_1, \dots, x_n)$ on V, and $(y_1, \dots, y_m)$ on W, we may express f in terms of its m components $f_j = y_j \circ f$, and then the linear map $Df(p)$ may be written as an $m \times n$ matrix, called the Jacobian matrix *of f at p.*

$$Df(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \tag{8}$$

We say that f is differentiable on U when it is differentiable at all $p \in U$ and we say it is continuously differentiable when

$$Df : U \longrightarrow \mathrm{Hom}(V, W)$$

[1] A norm on a vector space V is a map $|\cdot| : V \longrightarrow \mathbb{R}$ such that $||av|| = |a|||v||$ for $a \in \mathbb{R}$, $||v|| = 0$ iff $v = 0$, and satisfying the triangle inequality.

is continuous. The vector space of continuously differentiable functions on U with values in W is called $C^1(U, W)$.

The first derivative Df is also a map from U to a vector space ($\mathrm{Hom}(V, W)$), therefore if its derivative exists, we obtain a map

$$D^2 f : U \longrightarrow \mathrm{Hom}(V, \mathrm{Hom}(V, W)),$$

and so on. The vector space of k times continuously differentiable functions on U with values in W is called $C^k(U, W)$. We are most interested in C^∞ or "smooth" maps, all of whose derivatives exist; the space of these is denoted $C^\infty(U, W)$, and hence we have

$$C^\infty(U, W) = \bigcap_k C^k(U, W).$$

Note: for a C^2 function, $D^2 f$ actually has values in a smaller subspace of $V^* \otimes V^* \otimes W$, namely in $S^2 V^* \otimes W$, since "mixed partials are equal".

After this aside, we can define a smooth manifold.

Definition 2. A *smooth manifold* is a topological manifold equipped with an equivalence class of smooth atlases, explained below.

Definition 3. An atlas $\mathcal{A} = \{U_i, \varphi_i\}$ for a topological manifold is called *smooth* when all gluing maps

$$\varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij})$$

are smooth maps, i.e. lie in $C^\infty(\varphi_i(U_{ij}), \mathbb{R}^n)$. Two atlases $\mathcal{A}, \mathcal{A}'$ are *equivalent* if $\mathcal{A} \cup \mathcal{A}'$ is itself a smooth atlas.

Note: Instead of requiring an atlas to be smooth, we could ask for it to be C^k, or real-analytic, or even holomorphic (this makes sense for a $2n$-dimensional topological manifold when we identify $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

We may now verify that all the examples from section 1.1 are actually smooth manifolds:

Example 1.8 (Circle)**.** *For Example 1.1, only two charts, e.g. $\varphi_{\pm 1}$, suffice to define an atlas, and we have*

$$\varphi_{-1} \circ \varphi_1^{-1} = \begin{cases} t + 1 & -\frac{1}{2} < t < 0 \\ t & 0 < t < \frac{1}{2}, \end{cases}$$

which is clearly C^∞. In fact all the charts φ_z are smoothly compatible. Hence the circle is a smooth manifold.

The Cartesian product of smooth manifolds inherits a natural smooth structure from taking the Cartesian product of smooth atlases. Hence the n-torus, for example, equipped with the atlas we described in Example 1.2, is smooth. Example 1.3 is clearly defining a smooth manifold, since the restriction of a smooth map to an open set is always smooth.

Example 1.9 (Spheres)**.** *The charts for the n-sphere given in Example 1.4 form a smooth atlas, since*

$$\varphi_N \circ \varphi_S^{-1} : \vec{z} \mapsto \tfrac{1-x_0}{1+x_0} \vec{z} = \tfrac{(1-x_0)^2}{|\vec{x}|^2} \vec{z} = |\vec{z}|^{-2} \vec{z},$$

which is smooth on $\mathbb{R}^n \backslash \{0\}$, as required.

Example 1.10 (Projective spaces)**.** *The charts for projective spaces given in Example 1.5 form a smooth atlas, since*

$$\varphi_1 \circ \varphi_0^{-1}(z_1, \ldots, z_n) = (z_1^{-1}, z_1^{-1} z_2, \ldots, z_1^{-1} z_n), \tag{9}$$

which is smooth on $\mathbb{R}^n \backslash \{z_1 = 0\}$, as required, and similarly for all φ_i, φ_j.

The connected sum in Example 1.6 is clearly smooth since ϕ was chosen to be a smooth map.