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Generalized Kähler Geometry

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Abstract: Generalized Kähler geometry is the natural analogue of Kähler geometry, in the context of generalized complex geometry. Just as we may require a complex structure to be compatible with a Riemannian metric in a way which gives rise to a symplectic form, we may require a generalized complex structure to be compatible with a metric so that it defines a second generalized complex structure. We prove that generalized Kähler geometry is equivalent to the bi-Hermitian geometry on the target of a 2-dimensional sigma model with (2, 2) supersymmetry. We also prove the existence of natural holomorphic Courant algebroids for each of the underlying complex structures, and that these split into a sum of transverse holomorphic Dirac structures. Finally, we explore the analogy between pre-quantum line bundles and gerbes in the context of generalized Kähler geometry.

Contents

Introduction

Generalized Kähler geometry is the natural Riemannian geometry associated to a generalized complex structure in the sense of Hitchin. Just as in Kähler geometry, which involves a complex structure compatible with a symplectic form, a generalized Kähler structure derives from a compatible pair of generalized complex structures. A fundamental feature of generalized geometry is that it occurs on a manifold equipped with a Courant algebroid, a structure characterized by a class in the third cohomology with real coefficients. If this class is integral, the Courant algebroid may be thought of as arising from a rank one abelian gerbe. We view this gerbe as the analogue of the prequantum line bundle in the geometric quantization of symplectic manifolds.

As in the Kähler case, the existence of a generalized Kähler structure places strong constraints on the underlying manifold; we shall see that it inherits a pair of usual complex structures (I_+, I_-), which need not be isomorphic as complex manifolds. Interestingly, generalized Kähler structures may exist on complex manifolds which admit no Kähler metric: if the background Courant algebroid has nonzero characteristic class, we shall see that the complex structures must fail to satisfy the $\partial\overline{\partial}$-lemma and hence cannot be algebraic or even Moishezon.

The first result of this paper is that generalized Kähler geometry is equivalent to a bi-Hermitian geometry discovered by Gates et al. [15] in 1984, which arises on the target of a 2-dimensional sigma model upon imposing $N = (2, 2)$ supersymmetry. This equivalence, obtained in the author's thesis [23], was followed by a number of results, such as those contained in [8,10,17,19,22,24,29,30,32,35], which are not easily accessed purely from the bi-Hermitian point of view. The proof presented here is more transparent than that in [23].

The second group of results concerns the relationship between the complex structures (I_+, I_-) participating in the bi-Hermitian pair. We show that each of these complex manifolds inherits a holomorphic Courant algebroid, which actually splits as a sum of transverse holomorphic Dirac structures. This situation fits into the formalism developed by Liu et al. in [37]. Having this structure on each complex manifold, we may describe the relationship between them as a Morita equivalence between a Dirac structure on I_+ and its counterpart on I_-. We also explain how to interpret these facts from the point of view of the prequantum gerbe, following and extending some of the work of Hull et al. [31,36,51] on the relation between gerbes and generalized Kähler geometry.

1. Generalized Kähler Geometry

In Kähler geometry, a complex structure I is required to be compatible with a Riemannian metric g in such a way that the two-form $\omega = gI$ defines a symplectic structure. The introduction of a Riemannian metric may be thought of as a reduction of structure for TM; the complex structure provides a $GL(n, \mathbb{C})$ structure which is then reduced by g to the compact Lie group $U(n)$.

A generalized complex structure on a real exact Courant algebroid E reduces the usual orthogonal structure $O((n, n), \mathbb{R})$ of this bundle to the split unitary group $U(n, n)$. Generalized Kähler geometry may be viewed as an integrable reduction of this structure to its maximal compact subgroup $U(n) \times U(n)$, by the choice of a compatible generalized metric.

1.1. Generalized complex and Dirac geometry. Recall from [37] that an exact Courant algebroid $(E, \pi, q, [\cdot,\cdot])$ is a vector bundle E which is an extension of the form

$$0 \longrightarrow T^*M \xrightarrow{\iota} E \xrightarrow{\pi} TM \longrightarrow 0\,, \tag{1.1}$$

with a symmetric pairing $\langle\cdot,\cdot\rangle := (q(\cdot))(\cdot)$ given by a self-dual isomorphism of extensions $q : E \longrightarrow E^*$, and a bracket $[\cdot,\cdot]$ on its sheaf of sections such that, locally, there is a splitting of π inducing an isomorphism with $\mathbb{T}M = TM \oplus T^*M$, equipped with its usual symmetric pairing and Courant–Dorfman bracket $[X+\xi, Y+\eta] = [X, Y] + L_X\eta - i_Y d\xi$. Until we reach §2, we will work only with *real* Courant algebroids.

Definition 1.1. *A generalized complex structure $\mathbb{J}$ is an orthogonal bundle endomorphism of E, such that $\mathbb{J}^2 = -1$, and whose +i eigenbundle $L \subset E \otimes \mathbb{C}$ is involutive.*

The endomorphism $\mathbb{J}$ may preserve the subbundle $T^*M \subset E$, in which case it induces a complex structure on the underlying manifold. In general, $\mathbb{J}$ is not required to preserve the structure of E as an extension; indeed $\mathbb{J}(T^*M)$ may be disjoint from T^*M, in which case $\mathbb{J}(T^*M)$ provides a splitting of $\pi : E \longrightarrow TM$ with isotropic and involutive image, and therefore an isomorphism $E \cong \mathbb{T}M$, with $\mathbb{J}$ necessarily of the form

$$\mathbb{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \tag{1.2}$$

for $\omega : TM \longrightarrow T^*M$ a symplectic form. In general, $\mathbb{J}(T^*M)$ is a maximal isotropic, involutive subbundle (a Dirac structure) whose intersection with T^*M may vary in rank over the manifold. Indeed, $Q = \pi \circ \mathbb{J}|_{T^*M} : T^*M \longrightarrow TM$ is a real Poisson structure controlling the local behaviour of the geometry, in the sense that near a regular point of Q, $\mathbb{J}$ is isomorphic to the product of a complex and a symplectic structure [25].

Example 1.2. A holomorphic Poisson bivector field σ on the complex manifold (M, I) determines the following generalized complex structure on the standard Courant algebroid $E = \mathbb{T}M$:

$$\mathbb{J}_\sigma := \begin{pmatrix} I & Q \\ 0 & -I^* \end{pmatrix}, \tag{1.3}$$

where Q is the imaginary part of σ. The peculiar aspect of the generalized complex structure $\mathbb{J}_\sigma$ is that the complex structure obtained from its action on $TM \subset \mathbb{T}M$ is not intrinsic. Indeed, a symmetry of the Courant algebroid, such as is given by a closed 2-form $B \in \Omega^{2,\mathrm{cl}}(M)$, conjugates (1.3) into

$$\mathrm{e}^B \mathbb{J}_\sigma \mathrm{e}^{-B} = \begin{pmatrix} I - QB & Q \\ I^*B + BI - BQB & BQ - I^* \end{pmatrix}. \tag{1.4}$$

In fact, in [24] it is shown that in some cases, B may be chosen so that $I^*B + BI - BQB$ vanishes, rendering (1.4) again into the form (1.3), but for a different complex structure $J = I - QB$.

The $\pm i$ eigenbundles $L, \overline{L} \subset E \otimes \mathbb{C}$ of a generalized complex structure define two Dirac structures which are transverse, i.e. $L \cap \overline{L} = \{0\}$. In such a situation, as shown in [37], the pair of Lie algebroids satisfy a compatibility condition rendering (L, L^*) a Lie bialgebroid. Identifying $\overline{L}$ with L^* using the symmetric pairing, this means that the Lie bracket on $\overline{L}$ may be extended to a Schouten bracket on the sheaf of graded algebras $\Omega^\bullet_L = \Gamma^\infty(\wedge^k L^*)$, and that the Lie algebroid differential d_L on this algebra is a graded derivation of the bracket. In summary, we obtain a sheaf of differential graded Lie algebras from the transverse Dirac structures $(L, \overline{L})$.

$$(L, \overline{L}) \rightsquigarrow (\Omega^\bullet_L, d_L, [\cdot,\cdot]) \tag{1.5}$$

The above differential graded Lie algebra controls the elliptic deformation theory of generalized complex structures [25].

Transverse Dirac structures such as $(L, \overline{L})$ enjoy a further algebraic compatibility condition, as follows. We first recall the Baer sum operation on Courant algebroids [6,25,46], which is a realization of the additive structure on Ševera classes in $H^1(\Omega^{2,\mathrm{cl}})$.

Definition 1.3. *Given two Courant algebroids E_1, E_2 on M with projections $\pi_i : E_i \mapsto TM$, their Baer sum as extensions of TM by T^*M, namely the bundle*

$$E_1 \boxtimes E_2 := (E_1 \oplus_{TM} E_2)/K$$

for $K = \{(-\pi_1^*\xi, \pi_2^*\xi) : \xi \in T^*\}$, carries a natural Courant algebroid structure, defined componentwise.

The standard Courant algebroid on $\mathbb{T}M$ is the identity element for the Baer sum, and the inverse of a Courant algebroid $(E, \pi, q, [\cdot,\cdot])$, called the transpose $E^\top$, is given simply by reversing the symmetric inner product: $E^\top = (E, \pi, -q, [\cdot,\cdot])$. The Baer sum operation may also be applied to Dirac structures, as explained in [2,25]:

Proposition 1.4. *If $D_1 \subset E_1$ and $D_2 \subset E_2$ are Dirac structures which are transverse over TM, in the sense that $\pi_1(D_1) + \pi_2(D_2) = TM$, then their Baer sum*

$$D_1 \boxtimes D_2 := \frac{(D_1 \oplus_{TM} D_2) + K}{K},$$

for K as in Definition 1.3, is a Dirac structure in $E_1 \boxtimes E_2$.

A pair of Dirac structures D_1, D_2 such that $D_1 \cap D_2 = \{0\}$ are transverse in the above sense, and we may form their Baer sum $D_1^\top \boxtimes D_2 \subset \mathbb{T}M$, where $D_1^\top$ is simply D_1 viewed as a Dirac structure in $E^\top$. From observations made in [39], it follows that this Baer sum is given by

$$D_1^\top \boxtimes D_2 = \Gamma_\beta \subset \mathbb{T}M, \tag{1.6}$$

where Γ_β is the graph of a Poisson structure β (see [2] for a proof). Recall that, as a Lie algebroid, $\Gamma_\beta \cong T^*M$ has bracket given by $[df, dg] = d(\beta(df, dg))$, and anchor map $\beta : T^*M \longrightarrow TM$. The Poisson structure β may also be described in the following way: let $P_{D_i} : E \longrightarrow D_i$ be the projection operators for the direct sum $E = D_1 \oplus D_2$. Then β is given by

$$\beta = \pi \circ P_{D_1}|_{T^*M} : T^*M \longrightarrow TM. \tag{1.7}$$

The geometry induced by such a pair of transverse Dirac structures (D_1, D_2) in E may be understood in the following way: by projection to TM, D_1 and D_2 each induce singular foliations $\mathcal{F}_1, \mathcal{F}_2$ on the manifold M. The transversality condition on the D_i implies that the induced foliations are transverse, in the sense $T\mathcal{F}_1 + T\mathcal{F}_2 = TM$. As shown in [25], the exact Courant algebroid E may be pulled back to any submanifold $\iota : S \hookrightarrow M$, yielding an exact Courant algebroid over S, defined by

$$E_S := K^{\perp}/K, \tag{1.8}$$

where $K = N^*S \subset E|_S$ is the conormal bundle of S. If S happens to be a leaf of the singular foliation $\mathcal{F}$ induced by a Dirac structure $D \subset E$, then the Dirac structure also pulls back, yielding an isotropic, involutive splitting s_D of $\pi : E_S \longrightarrow TS$. Therefore, in the presence of two transverse Dirac structures (D_1, D_2), if we pull back E to a leaf S of the singular foliation induced by $D_1^{\top} \boxtimes D_2$, it will have two splittings s_{D_1}, s_{D_2}, each obtained from one of the Dirac structures. The resulting splittings are themselves transverse in E_S, and therefore they differ by a section $\omega_S \in \Omega^{2,\mathrm{cl}}(S)$ which must be nondegenerate. This is precisely the symplectic form on the leaf of the Poisson structure β.

Finally, we emphasize an algebraic implication of the Baer sum identity described above.

Proposition 1.5. *Let $D_1, D_2 \subset E$ be Dirac structures such that $D_1 \cap D_2 = \{0\}$. Then the Baer sum $D_1^{\top} \boxtimes D_2 = \Gamma_\beta$, coincides with the fiber product of the Lie algebroids D_1, D_2 over TM. As a result, we have the isomorphism of sheaves of differential graded algebras*

$$(\wedge^\bullet \mathcal{T}_M, d_\beta) = (\Omega^\bullet_{D_1}, d_{D_1}) \otimes_{\Omega^\bullet_T} (\Omega^\bullet_{D_2}, d_{D_2}), \tag{1.9}$$

where $(\wedge^\bullet \mathcal{T}_M, d_\beta = [\beta, \cdot])$ is the Lichnerowicz complex[1] *of sheaves of multivector fields induced by the Schouten bracket with the Poisson structure β, and the anchor maps $\pi_i : D_i \longrightarrow TM$ induce the morphisms $\pi_i^* : \Omega^\bullet_T \longrightarrow \Omega^\bullet_{D_i}$ from the usual de Rham complex of M, which are used in the tensor product.*

Proof. This follows from the simple observation that $K = \{(-\pi_1^*\xi, \pi_2^*\xi) : \xi \in T^*M\}$ intersects $D_1 \oplus_{TM} D_2 = \{(d_1, d_2) \in D_1 \oplus D_2 : \pi_1(d_1) = \pi_2(d_2)\}$ precisely in $D_1 \cap D_2 \cap T^*M$, which vanishes since $D_1 \cap D_2 = \{0\}$. In this way, $D_1^{\top} \boxtimes D_2$ coincides, as a Lie algebroid, with the fiber product of D_1, D_2 as Lie algebroids, yielding the diagram of Lie algebroids

$$\begin{array}{ccc} \Gamma_\beta & \longrightarrow & D_2 \\ \downarrow & & \downarrow \\ D_1 & \longrightarrow & TM \end{array}$$

which dualizes to the fact that the Lichnerowicz complex is given by the (graded) tensor product (1.9). □

This is of particular importance when studying modules over the Lie algebroids D_1 or D_2, i.e. vector bundles (or sheaves of $\mathcal{O}_M$-modules) with flat D_i-connections.

Corollary 1.6. *For a pair of transverse Dirac structures (D_1, D_2), the tensor product of a D_1-module and a D_2-module is a Γ_β-module, i.e. a Poisson module [41,43]. In particular, any D_i-module is also a Poisson module.*

[1] The hypercohomology of this complex is the well-known Poisson cohomology of β.

In the case of a generalized complex structure, we have the transverse Dirac structures $(L, \overline{L})$, and it was shown in [25] that their Baer sum is

$$L^\top \boxtimes \overline{L} = \Gamma_{iQ/2} \subset \mathbb{T}M, \tag{1.10}$$

for the Poisson structure $Q = \pi \circ \mathbb{J}|_{T^*M}$ described earlier. A vector bundle with a flat $\overline{L}$-connection on a generalized complex manifold is called a generalized holomorphic bundle [25], hence we have the following consequence of the above Baer sum:

Corollary 1.7. *Any generalized holomorphic bundle inherits a Poisson module structure, for the underlying real Poisson structure.*

1.2. Generalized Kähler structures. Let $(E, \pi, q, [\cdot, \cdot])$ be an exact real Courant algebroid over the smooth manifold M.

Definition 1.8. *A generalized Kähler structure on E is a pair $(\mathbb{J}_+, \mathbb{J}_-)$ of generalized complex structures on E which commute, i.e. $\mathbb{J}_+\mathbb{J}_- = \mathbb{J}_-\mathbb{J}_+$, and such that the symmetric pairing*

$$G(x, y) := \langle \mathbb{J}_+ x, \mathbb{J}_- y\rangle \tag{1.11}$$

is positive-definite, defining a metric on E called the generalized Kähler metric.

A usual Kähler structure on a manifold is given by a complex structure I compatible with a Riemannian metric g, in the sense that $\omega := gI$ is a symplectic form. This defines the following generalized Kähler structure on $\mathbb{T}M = TM \oplus T^*M$:

$$\mathbb{J}_+ = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \quad \mathbb{J}_- = \begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix}, \tag{1.12}$$

so that $G(X + \xi, Y + \eta) = \frac{1}{2}(g(X, Y) + g^{-1}(\xi, \eta))$ is the usual Kähler metric. The generalized Kähler metric (1.11) is an example of a generalized metric, which may be viewed as a reduction of structure for the Courant algebroid E, from its usual $O(n, n)$ structure to the compact form $O(n) \times O(n)$.

Definition 1.9. *A generalized metric G on E is a positive-definite metric on E which is compatible with the pre-existing symmetric pairing $\langle \cdot, \cdot \rangle$, in the sense that it is obtained by choosing a maximal positive-definite subbundle $C_+ \subset E$ (with orthogonal complement $C_- := C_+^\perp$), and defining*

$$G(x, y) := \langle x_+, y_+\rangle - \langle x_-, y_-\rangle,$$

where $x_\pm$ denotes the orthogonal projection to $C_\pm$.

Identifying E with E^* using $\langle \cdot, \cdot \rangle$, we may view G as a self-adjoint endomorphism $G : E \longrightarrow E$, with ± 1 eigenbundle given by $C_\pm$. For a generalized Kähler structure, $G = -\mathbb{J}_+\mathbb{J}_-$, so that for (1.12), G is given by

$$G = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}.$$

Example 1.10. Any positive-definite subbundle $C_+ \subset \mathbb{T}M$ is the graph of a bundle map $\theta : TM \longrightarrow T^*M$ with positive-definite symmetric part. That is, $\theta = b + g$, with $b \in \Omega^2(M)$ and g a Riemannian metric. Then $C_\pm$ is the graph of $b \pm g$. The corresponding endomorphism $G : E \longrightarrow E$ is then described by

$$G_{g,b} = e^b \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} e^{-b} = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix}.$$

Note that the induced Riemannian metric on $TM \subset \mathbb{T}M$ is $g - bg^{-1}b$, while the metric on T^*M is the usual inverse metric of g.

An immediate consequence of the choice of generalized metric is that the projection $\pi : E \longrightarrow TM$ has two splittings $s_\pm$ corresponding to the two subbundles $C_\pm$. The average of these splittings, $s = \frac{1}{2}(s_+ + s_-)$, is then a splitting of π with isotropic image $G(T^*M)$. The splitting induces an isomorphism $s_* : E \longrightarrow \mathbb{T}M$ which sends the definite subbundles $C_\pm$ to the graphs $\Gamma_{\pm g}$, for g a Riemannian metric on M. In summary, we have the following:

Proposition 1.11. *The choice of a generalized metric G on E is equivalent to a choice of isotropic splitting $s : TM \longrightarrow E$, together with a Riemannian metric g on M, such that*

$$C_\pm = \{s(X) \pm g(X) \ : \ X \in TM\}. \tag{1.13}$$

As a result of the splitting s determined by the generalized metric, we apply the results of [46] to immediately obtain a closed 3-form

$$H(X, Y, Z) := \langle s(X), [s(Y), s(Z)]\rangle, \tag{1.14}$$

called the *torsion* of the generalized metric. The torsion determines the Courant bracket induced on $\mathbb{T}M$ by s_*, via

$$[X + \xi, Y + \eta] = [X, Y] + L_X\eta - i_Y d\xi + i_X i_Y H. \tag{1.15}$$

We now describe the geometric structures induced on M by the generalized Kähler pair $(\mathbb{J}_+, \mathbb{J}_-)$. First, we leave aside questions of integrability and describe *almost* generalized Kähler structures, which are generalized Kähler structures without the Courant involutivity conditions on $\mathbb{J}_+$ and $\mathbb{J}_-$.

An almost generalized complex structure $\mathbb{J}_+$ is compatible with G when it preserves C_+ (and hence, necessarily, C_-), or equivalently, when it commutes with G. This compatibility is also equivalent to the fact that $\mathbb{J}_- := G\mathbb{J}_+$ is an almost generalized complex structure. Since $C_\pm$ are the ± 1 eigenbundles of G, we have

$$\mathbb{J}_+|_{C_\pm} = \pm\mathbb{J}_-|_{C_\pm}, \tag{1.16}$$

and so the complex structures on the bundles $C_\pm$ induced by $\mathbb{J}_+, \mathbb{J}_-$ coincide up to sign. Using the identifications of metric bundles

$$s_\pm : (TM, g) \longrightarrow (C_\pm, \pm\langle\cdot,\cdot\rangle),$$

we obtain two almost complex structures I_+, I_- on the manifold M, each of which is compatible with the Riemannian metric g, hence forming an almost bi-Hermitian structure. We now show that the correspondence $(\mathbb{J}_+, \mathbb{J}_-) \mapsto (g, I_+, I_-)$ is an equivalence.

Theorem 1.12. *An almost generalized Kähler structure* $(\mathbb{J}_+, \mathbb{J}_-)$ *on* E *is equivalent to the data* (s, g, I_+, I_-)*, where* s *is an isotropic splitting of* E*,* g *is a Riemannian metric, and* $I_\pm$ *are almost complex structures compatible with* g*.*

Proof. We have already explained how to extract $(s, g, I_\pm)$ from $(\mathbb{J}_+, \mathbb{J}_-)$. To reconstruct $(\mathbb{J}_+, \mathbb{J}_-)$ from the bi-Hermitian data, we construct definite splittings $s_\pm$ via

$$s_\pm := s \pm g : TM \longrightarrow E,$$

following Eq. (1.13), and use the fact that $\mathbb{J}_+, \mathbb{J}_-$ are built from the complex structures $I_\pm$ by transporting them to $C_\pm$ and using Eq. (1.16):

$$\mathbb{J}_\pm := s_+ I_+ s_+|_{C_+}^{-1} \pm s_- I_- s_-|_{C_-}^{-1}, \tag{1.17}$$

which expands to the expression:

$$\mathbb{J}_\pm = s_*^{-1} \frac{1}{2} \begin{pmatrix} I_+ \pm I_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(I_+^* \pm I_-^*) \end{pmatrix} s_*, \tag{1.18}$$

where $s_* : E \longrightarrow \mathbb{T}M$ is the isomorphism induced by s, and $\omega_\pm := gI_\pm$ are the nondegenerate 2-forms determined by the almost Hermitian structures $(g, I_\pm)$. The two constructions are easily seen to be mutually inverse. □

Before proceeding to translate the integrability condition from the generalized complex structures to the bi-Hermitian data, we make some comments concerning orientation.

Remark 1.13. The *type* [25] of a generalized complex structure $\mathbb{J}$ at a point is defined to be

$$\text{type}(\mathbb{J}) := \frac{1}{2}\text{corank}_{\mathbb{R}}(Q),$$

where $Q = \pi \circ \mathbb{J}|_{T^*}$ is the real Poisson structure associated to $\mathbb{J}$. On a real $2n$-manifold, the type may vary between 0, where $\mathbb{J}$ defines a symplectic structure, and n, where it defines a complex structure. The parity of the type, however, is locally constant, as it is determined by the orientation induced by $\mathbb{J}$ on E ($\det E$ is canonically trivial, we may view $\mathbb{J}$ as a section of $\wedge^2 E$, and $\mathbb{J}^{2n}/(2n)! = +1$ or -1 as type($\mathbb{J}$) is even or odd, respectively). If we have an almost generalized Kähler structure on a real $2n$-manifold, the equation $G = -\mathbb{J}_+\mathbb{J}_-$ yields

$$\frac{1}{(2n)!}\mathbb{J}_+^{2n} \frac{1}{(2n)!}\mathbb{J}_-^{2n} = (-1)^n,$$

Implying that $\mathbb{J}_+$ and $\mathbb{J}_-$ must have equal parity in real dimension $4k$ and unequal parity in real dimension $4k + 2$. Furthermore, by Eq. (1.17), the parity of $\mathbb{J}_+$ is even or odd as the orientations induced by $I_\pm$ agree or disagree, respectively. This leads immediately to the fact that in real dimension $4k$, both $\mathbb{J}_\pm$ may either have even parity, in which case $I_\pm$ induce the same orientation, or odd parity, in which case $I_\pm$ induce opposite orientations on M. In dimension $4k + 2$, however, there is no constraint placed on the orientations of $I_\pm$, since I_+ may be replaced with $-I_+$ without altering the parity of $\mathbb{J}_\pm$.

Example 1.14. If $\dim_{\mathbb{R}} M = 4$, an almost generalized Kähler structure may either have $\text{type}(\mathbb{J}_+) = \text{type}(\mathbb{J}_-) = 1$, in which case $I_\pm$ induce opposite orientations, or $\mathbb{J}_\pm$ both have even type, in which case $I_\pm$ must induce the same orientation on M.

1.3. Integrability and bi-Hermitian geometry. Let $(s, g, I_\pm)$ be the almost bi-Hermitian data corresponding to an almost generalized Kähler structure $(\mathbb{J}_+, \mathbb{J}_-)$, as in Theorem 1.12. In this section, we describe the integrability conditions on $(s, g, I_\pm)$ corresponding to the integrability of $\mathbb{J}_+$ and $\mathbb{J}_-$.

Recall that the integrability condition for $\mathbb{J}_\pm$ is that the $+i$-eigenbundles $L_\pm = \ker(\mathbb{J}_\pm - i\mathbf{1})$ are involutive for the Courant bracket on $E \otimes \mathbb{C}$. Since $\mathbb{J}_\pm$ commute, the eigenbundle of $\mathbb{J}_+$ decomposes into eigenbundles of $\mathbb{J}_-$, so that

$$L_+ = \ell_+ \oplus \ell_-, \tag{1.19}$$

where $\ell_+ = L_+ \cap L_-$ and $\ell_- = L_+ \cap \overline{L_-}$. Since $G = -\mathbb{J}_+\mathbb{J}_-$ has eigenvalue +1 on $\ell_+ \oplus \overline{\ell}_+$, we also have

$$C_\pm \otimes \mathbb{C} = \ell_\pm \oplus \overline{\ell}_\pm. \tag{1.20}$$

As a result, we obtain a decomposition of the Courant algebroid into four isotropic subbundles, each of complex dimension n on a real $2n$-manifold:

$$E \otimes \mathbb{C} = \ell_+ \oplus \ell_- \oplus \overline{\ell}_+ \oplus \overline{\ell}_-. \tag{1.21}$$

Since $\ell_\pm$ are intersections of involutive subbundles, they are individually involutive. This is actually an equivalent characterization of the integrability condition on $\mathbb{J}_\pm$.

Proposition 1.15. *The almost generalized Kähler structure* $(\mathbb{J}_+, \mathbb{J}_-)$ *is integrable if and only if both the subbundles* $\ell_\pm$ *described above are involutive.*

Proof. That the integrability of $\mathbb{J}_\pm$ implies the involutivity of $\ell_\pm$ is explained above. Now let $\ell_\pm$ be involutive. We must show that $L_+ = \ell_+ \oplus \ell_-$ and $L_- = \ell_+ \oplus \overline{\ell}_-$ are involutive. To prove that L_+ is involutive, we need only show that if $x_\pm$ is a section of $\ell_\pm$, then $[x_+, x_-]$ is a section of L_+. We do this by showing $[x_+, x_-]$ is orthogonal to both ℓ_+, ℓ_-, and hence must lie in $\ell_+^\perp \cap \ell_-^\perp = (\ell_+ \oplus \ell_-)^\perp = L_+^\perp = L_+$, where we have used the fact that L_+ is maximal isotropic. For $y_\pm$ any section of $\ell_\pm$ we have:

$$\begin{aligned}
\langle [x_+, x_-], y_+\rangle &= \pi(x_+)\langle x_-, y_+\rangle - \langle x_-, [x_+, y_+]\rangle = 0,\\
\langle [x_+, x_-], y_-\rangle &= -\langle [x_-, x_+], y_-\rangle = \pi(x_-)\langle x_+, y_-\rangle - \langle x_+, [x_-, y_-]\rangle = 0,
\end{aligned}$$

hence $[x_+, x_-]$ is in L_+, as required. L_- is shown to be involutive in the same way. □

To understand what this integrability condition imposes on the bi-Hermitian data, we use Theorem 1.12 to express the bundles $\ell_\pm$ purely in terms of the data $(s, g, I_\pm)$. By (1.19) and (1.20), we see that $\ell_\pm$ is the $+i$ eigenbundle of $\mathbb{J}_+$ acting on $C_\pm \otimes \mathbb{C}$. Since the almost complex structures $I_\pm$ are defined via the restriction of $\mathbb{J}_+$ to $C_\pm$, we obtain:

$$\ell_\pm = \{(s \pm g)X \ : \ X \in T_\pm^{1,0}M\}, \tag{1.22}$$

where $T_\pm^{1,0}M$ is the $+i$ eigenbundle of the almost complex structure $I_\pm$. Using the 2-forms $\omega_\pm = gI_\pm$, we obtain a more useful form of Eq. (1.22):

$$\begin{aligned}
\ell_\pm &= \{sX \mp i\omega_\pm X \ : \ X \in T_\pm^{1,0}M\}\\
&= \mathrm{e}^{\mp i\omega_\pm} s(T_\pm^{1,0}M),
\end{aligned} \tag{1.23}$$

where $\mathrm{e}^{\mp i\omega_\pm}$ acts on $x \in E$ via $x \mapsto x + i_{\pi(x)}(\mp i\omega_\pm)$.

Theorem 1.16. *Let* $(\mathbb{J}_+, \mathbb{J}_-)$ *be an almost generalized Kähler structure, described equivalently by the almost bi-Hermitian data* (s, g, I_+, I_-) *as above.* $(\mathbb{J}_+, \mathbb{J}_-)$ *is integrable if and only if* $I_\pm$ *are integrable complex structures on* M*, and the following constraint holds:*

$$\pm\, d^c_\pm \omega_\pm = H, \tag{1.24}$$

where $H \in \Omega^{3,\mathrm{cl}}(M, \mathbb{R})$ *is the closed 3-form corresponding to the section* s *via Eq. (1.14), and* $d^c_\pm = i(\overline{\partial}_\pm - \partial_\pm)$ *are the real Dolbeault operators corresponding to the complex structures* $I_\pm$.

Proof. Using expression (1.23) for $\ell_\pm$, let $\mathrm{e}^{\mp i\omega_\pm}s(X)$, $\mathrm{e}^{\mp i\omega_\pm}s(Y)$ be two sections of $\ell_\pm$, where X, Y are vector fields in $T^{1,0}_\pm M$. Then the properties of the Courant bracket and the definition of H from Eq. (1.14) yield

$$\begin{aligned}[\mathrm{e}^{\mp i\omega_\pm}s(X), \mathrm{e}^{\mp i\omega_\pm}s(Y)] &= \mathrm{e}^{\mp i\omega_\pm}[s(X), s(Y)] + i_Y i_X d(\mp i\omega_\pm)\\ &= \mathrm{e}^{\mp i\omega_\pm}(s([X,Y]) + i_X i_Y H) + i_Y i_X d(\mp i\omega_\pm)\\ &= \mathrm{e}^{\mp i\omega_\pm}s([X,Y]) + i_X i_Y (H \pm i d\omega_\pm).\end{aligned}$$

This is again a section of $\ell_\pm$ if and only if $[X, Y]$ is in $T^{1,0}_\pm M$ and $(H \pm i d\omega_\pm)^{(3,0)+(2,1)}$ vanishes. The first condition is precisely the integrability of the complex structures $I_\pm$, and in this case since $\omega_\pm$ is of type $(1, 1)$ with respect to $I_\pm$, $d\omega_\pm$ has no $(3, 0)$ component. The second condition is then the statement that

$$H^{2,1} = \mp i\partial_\pm\omega_\pm,$$

which together with its complex conjugate yields $H = \pm d^c_\pm\omega_\pm$, as required. □

The above theorem demonstrates that generalized Kähler geometry, involving a pair of commuting generalized complex structures, may be viewed classically as a bi-Hermitian geometry, in which the pair of usual complex structures need not commute, and with an additional constraint involving the torsion 3-form H. This bi-Hermitian geometry is known in the physics literature: Gates et al. showed in [15] that upon imposing $N = (2, 2)$ supersymmetry, the geometry induced on the target of a 2-dimensional sigma model is precisely this one.

Corollary 1.17. *If the torsion* H *of a compact generalized Kähler manifold has nonvanishing cohomology class in* $H^3(M, \mathbb{R})$*, then the complex structures* $I_\pm$ *must both fail to satisfy the* $dd^c_\pm$*-lemma; in particular, they do not admit Kähler metrics and are not algebraic varieties.*

Proof. Suppose I_+ satisfies the dd^c_+-lemma. Then since $H = d^c\omega_+$ and $dH = 0$, we conclude there exists $a_+ \in \Omega^1(M, \mathbb{R})$ with $H = dd^c_+ a_+$, implying H is exact. The same argument holds for I_-. □

1.4. Examples of generalized Kähler manifolds. The main examples of generalized Kähler manifolds in the literature were constructed in several different ways: by imposing symmetry [29], by a generalized Kähler reduction procedure analogous to symplectic reduction [8,9,35], by recourse to twistor-theoretic results on surfaces [3,4], by a flow construction using the underlying real Poisson geometry [24,30], and by developing a deformation theory for generalized Kähler structures [17–20] whereby one may deform usual Kähler structures into generalized ones. We elaborate on some illustrative examples from [23].

Example 1.18. Let (M, g, I, J, K) be a hyper-Kähler structure. Then clearly (g, I, J) is a bi-Hermitian structure, and since $d\omega_I = d\omega_J = 0$, we see that (g, I, J) defines a generalized Kähler structure for the standard Courant structure on $\mathbb{T}M$. From formula (1.18), we reconstruct the generalized complex structures:

$$\mathbb{J}_\pm = \frac{1}{2}\begin{pmatrix} I \pm J & -(\omega_I^{-1} \mp \omega_J^{-1}) \\ \omega_I \mp \omega_J & -(I^* \pm J^*) \end{pmatrix}. \tag{1.25}$$

Note that (1.25) describes two generalized complex structures of symplectic type, a fact made manifest via the following expression:

$$\mathbb{J}_\pm = \mathrm{e}^{\pm\omega_K}\begin{pmatrix} 0 & -\frac{1}{2}(\omega_I^{-1} \mp \omega_J^{-1}) \\ \omega_I \mp \omega_J & 0 \end{pmatrix}\mathrm{e}^{\mp\omega_K}.$$

The same observation holds for any two non-opposite complex structures I_1, I_2 in the 2-sphere of hyper-Kähler complex structures, namely that the bi-Hermitian structure given by (g, I_1, I_2) defines a generalized Kähler structure where both generalized complex structures are of symplectic type.

The bi-Hermitian structure obtained from a hyperkähler structure is an example of a *strongly* bi-Hermitian structure in the sense of [3], i.e. a bi-Hermitian structure such that I_+ is nowhere equal to $\pm I_-$. From expression (1.18), it is clear that in four dimensions, strongly bi-Hermitian structures with equal orientation correspond exactly to generalized Kähler structures where both generalized complex structures are of symplectic type.

Example 1.19. The generalized Kähler structure described in Example 1.18 can be deformed using a method similar to that described in [24]. The complex 2-form $\sigma_I = \omega_J + i\omega_K$ on a hyper-Kähler structure is a holomorphic symplectic form with respect to I, and similarly $\sigma_J = -\omega_I + i\omega_K$ is holomorphic symplectic with respect to J. As in Eq. (1.3), these define generalized complex structures on $\mathbb{T}M$ given by:

$$\mathbb{J}_{\sigma_I} := \begin{pmatrix} I & \omega_K^{-1} \\ 0 & -I^* \end{pmatrix}, \quad \mathbb{J}_{\sigma_J} := \begin{pmatrix} J & \omega_K^{-1} \\ 0 & -J^* \end{pmatrix}. \tag{1.26}$$

Interestingly, the symmetry e^F, for the closed 2-form $F = \omega_I + \omega_J$, takes $\mathbb{J}_{\sigma_I}$ to $\mathbb{J}_{\sigma_J}$, so that

$$\mathrm{e}^F \mathbb{J}_{\sigma_I}\mathrm{e}^{-F} = \mathbb{J}_{\sigma_J}.$$

Now choose $f \in C^\infty(M, \mathbb{R})$ and let X_f be its Hamiltonian vector field for the Poisson structure ω_K^{-1}. Let φ_t be the flow generated by this vector field, and define

$$F_t(f) := \int_0^t \varphi_s^*(dd_J^c f)ds.$$

In [24], it is shown that the symmetry e^{F_t} takes $\mathbb{J}_{\sigma_J}$ to the deformed generalized complex structure

$$\mathbb{J}_{\sigma_{J_t}} = \begin{pmatrix} J_t & \omega_K^{-1} \\ 0 & -J_t^* \end{pmatrix},$$

where $J_t = \varphi_t^*(J)$, and that as a result, I and J_t give a family of generalized Kähler structures with respect to the deformed metric

$$g_t = -\frac{1}{2}(F + F_t(f))(I + J_t),$$

where $g_0 = -\frac{1}{2}F(I + J)$ is the original hyper-Kähler metric.

The idea of deforming a Hyperkähler structure to obtain a bi-Hermitian structure appeared, with a different formulation, in [3] (see also [29]), where it is shown for surfaces that the Hamiltonian vector field can be chosen so that the resulting deformed metric is not anti-self-dual, and hence by a result in [42], cannot admit more than two distinct orthogonal complex structures.

Example 1.20 (The Hopf surface: odd generalized Kähler). Consider the standard Hopf surface $X = (\mathbb{C}^2 - \{0\})/(x \mapsto 2x)$, and denote its complex structure by I_-. The product metric on $X \cong S^3 \times S^1$ can be written as follows in the usual affine coordinates on $\mathbb{C}^2$:

$$g = \frac{1}{4\pi R^2}(dx_1 d\bar{x}_1 + dx_2 d\bar{x}_2), \tag{1.27}$$

for $R^2 = x_1\bar{x}_1 + x_2\bar{x}_2$. The complex structure I_- is manifestly Hermitian for this metric, and has associated 2-form $\omega_- = gI_-$ given by:

$$\omega_- = \frac{i}{4\pi R^2}(dx_1 \wedge d\bar{x}_1 + dx_2 \wedge d\bar{x}_2).$$

The complex derivative $H = -d^c\omega_-$ is a real closed 3-form on X generating $H^3(X, \mathbb{Z})$.

Now let I_+ be the complex structure on the Hopf surface obtained by modifying the complex structure on $\mathbb{C}^2$ such that $(x_1, \overline{x_2})$ are holomorphic coordinates; note that $I_\pm$ have opposite orientations, and are both Hermitian with respect to g. Also, it is clear that $d_+^c\omega_+ = -d_-^c\omega_- = H$. Therefore, the bi-Hermitian data $(g, I_\pm)$ defines a generalized Kähler structure on $(\mathbb{T}X, H)$. Since $I_\pm$ induce opposite orientations, the corresponding generalized complex structures $\mathbb{J}_\pm$ are both of odd type, by Example 1.14. Note that the complex structures $I_\pm$ happen to commute, a special case studied in [4]. This particular generalized Kähler geometry first appeared in the context of a supersymmetric $SU(2) \times U(1)$ Wess–Zumino–Witten model [44].

Example 1.21 (The Hopf surface: even generalized Kähler). Let (g, I_-) be the standard Hermitian structure on the Hopf surface, as in Example 1.20. We specify a different complex structure I_+ by providing a generator $\Omega_+ \in \Omega_+^{2,0}(X)$, namely:

$$\Omega_+ := \frac{1}{R^4}(\bar{x}_1 dx_1 + x_2 d\bar{x}_2) \wedge (\bar{x}_1 dx_2 - x_2 d\bar{x}_1). \tag{1.28}$$

Comparing this with the usual complex structure, where the generator is given by $\Omega_- = \frac{1}{R^2}dx_1 \wedge dx_2$, we see that I_+ coincides with I_- along the curve $E_2 = \{x_2 = 0\}$, and coincides with $-I_-$ along $E_1 = \{x_1 = 0\}$. From the expression (1.28), we see that Ω_+ spans an isotropic plane for the metric (1.27), hence (g, I_+) is also Hermitian, with associated 2-form

$$\omega_+ = \frac{i}{4\pi R^2}(\theta_1 \wedge \bar{\theta}_1 + \theta_2 \wedge \bar{\theta}_2),$$

with $\theta_1 = \bar{x}_1 dx_1 + x_2 d\bar{x}_2$ and $\theta_2 = \bar{x}_1 dx_2 - x_2 d\bar{x}_1$. This 2-form also satisfies $d^c_+\omega_+ = H$, so that $(g, I_\pm)$ is an even generalized Kähler structure for $(\mathbb{T}X, H)$. From the explicit formulae (1.18) for $\mathbb{J}_\pm$, we see that their real Poisson structures are given by

$$Q_\pm = -\frac{1}{2}(\omega_+^{-1} \mp \omega_-^{-1}) = \frac{1}{2}(I_+ \mp I_-)g^{-1}. \tag{1.29}$$

Hence Q_+ drops rank from 4 to 0 along E_2, and Q_- drops rank similarly on E_1. In other words, $\mathbb{J}_\pm$ are generically of symplectic type but each undergoes type change to complex type along one of the curves.

The existence of generalized Kähler structures with nonzero torsion class on the Hopf surface implies, by Corollary 1.17, the well-known fact that the Hopf surface is non-algebraic. It is natural to ask whether the Hopf surface might admit generalized Kähler structures with vanishing torsion class. We now show that this is not the case.

Proposition 1.22. *Any generalized Kähler structure with I_+ given by the Hopf surface $X = (\mathbb{C}^2 - \{0\})/(x \mapsto 2x)$ must have nonvanishing torsion $[H] \in H^3(X, \mathbb{R})$.*

Proof. The Hopf surface has $h^{2,1} = 1$, generated by the (2,1) component of the standard volume form ν of S^3, which satisfies $[\nu^{2,1}] = [\nu^{1,2}] = \frac{1}{2}[\nu]$ in de Rham cohomology. Suppose that X were the I_+ complex structure in a generalized Kähler structure with torsion H. By the generalized Kähler condition, $H^{2,1} = -i\partial_+\omega_+$, and so $dH^{2,1} = -i\overline{\partial}_+\partial_+\omega_+ = 0$.

We claim that $H^{2,1}$ must be nonzero in Dolbeault cohomology. If not, we would have $\partial_+\omega_+ = \overline{\partial}_+\tau$, for $\tau \in \Omega^{2,0}(X)$. Now let $E = \{x_1 = 0\}$, a null-homologous holomorphic curve in X, and let D be a smooth 3-chain with $\partial D = E$. Then

$$\int_E \omega_+ = \int_D d\omega_+ = \int_D d(\tau + \bar{\tau}) = \int_E (\tau + \bar{\tau}),$$

which is a contradiction because ω_+ is a positive $(1,1)$ form, forcing the left hand side to be nonzero, while τ is of type $(2,0)$ and vanishes on E.

Because $h^{2,1} = 1$, there must exist $\sigma \in \Omega^{2,0}$ such that $H^{2,1} = c\nu^{2,1} + \overline{\partial}\sigma$, with $c \in \mathbb{C}^*$, and since $\overline{\partial}\sigma = d\sigma$, we have $[H^{2,1}] = c[\nu^{2,1}]$ in de Rham cohomology. But $[H^{2,1}] = \frac{1}{2}[H]$, since $H^{2,1} - H^{1,2} = -id\omega_+$. Hence $[H] \neq 0$ in $H^3(M, \mathbb{R})$. □

Example 1.23 (Even-dimensional real Lie groups). It has been known since the work of Samelson and Wang [45,49] that any even-dimensional real Lie group G admits left- and right-invariant complex structures J_L, J_R. If the group admits a bi-invariant positive-definite inner product $b(\cdot,\cdot)$, the complex structures can be chosen to be Hermitian with respect to b. The bi-Hermitian structure (b, J_L, J_R) is then a generalized Kähler structure on $(\mathbb{T}G, H)$, where H is the Cartan 3-form associated to b, defined by $H(X,Y,Z) = b([X,Y],Z)$. To see this, we compute $d^c_{J_L}\omega_{J_L}$:

$$\begin{aligned} A = d^c_{J_L}\omega_{J_L}(X,Y,Z) &= d\omega_{J_L}(J_L X, J_L Y, J_L Z) \\ &= -b([J_L X, J_L Y], Z) + c.p. \\ &= -b(J_L[J_L X, Y] + J_L[X, J_L Y] + [X,Y], Z) + c.p. \\ &= (2b([J_L X, J_L Y], Z) + c.p.) - 3H(X,Y,Z) \\ &= -2A - 3H(X,Y,Z), \end{aligned}$$

Proving that $d^c_{J_L}\omega_{J_L} = -H$. Since the right Lie algebra is anti-isomorphic to the left, the same calculation with J_R yields $d^c_{J_R}\omega_{J_R} = H$, and finally we have

$$-d^c_{J_L}\omega_{J_L} = d^c_{J_R}\omega_{J_R} = H,$$

as required. For $G = SU(2) \times U(1)$, we recover Example 1.21 from this construction. Note that J_L, J_R are isomorphic as complex manifolds, via the inversion on the group.

2. Holomorphic Dirac Geometry

In the previous section, we saw that a generalized Kähler structure on M gives rise to a pair of complex manifolds $X_\pm = (M, I_\pm)$ with the same underlying smooth manifold. In this section, we describe the relationship between the complex manifolds $X_\pm$. Although there is generally no morphism between X_+ and X_- in the holomorphic category, we show that $X_\pm$ are each equipped with holomorphic Courant algebroids which decompose as a sum of transverse holomorphic Dirac structures, and that the Dirac structures on X_+ are Morita equivalent to those on X_-. This provides a holomorphic interpretation to the deformation theory of generalized complex structures, as well as to the notion of a generalized holomorphic bundle.

2.1. Holomorphic Courant algebroids. In the following sections, we will need to understand Courant algebroids in the holomorphic category. Ševera's classification proceeds in the same way, with the result that the sheaf cohomology group $H^1(\Omega^{2,\mathrm{cl}})$ describes holomorphic Courant algebroids up to isomorphism. Here $\Omega^{k,\mathrm{cl}}$ refers to the sheaf of closed holomorphic differential k-forms. On a complex manifold, the following exact sequence of sheaves:

$$0 \longrightarrow \Omega^{2,\mathrm{cl}} \longrightarrow \Omega^2 \xrightarrow{\partial} \Omega^{3,\mathrm{cl}} \longrightarrow 0$$

leads to the long exact sequence:

$$H^0(\Omega^2) \xrightarrow{\partial} H^0(\Omega^{3,\mathrm{cl}}) \longrightarrow H^1(\Omega^{2,\mathrm{cl}}) \longrightarrow H^1(\Omega^2) \xrightarrow{\partial} H^1(\Omega^{3,\mathrm{cl}})\,. \quad (2.1)$$

If the $\partial\overline{\partial}$-lemma holds, then the left- and right-most maps in (2.1) vanish and $H^0(\Omega^{3,\mathrm{cl}}) = H^0(\Omega^3)$, exhibiting the classification of holomorphic Courant algebroids as an extension:

$$0 \longrightarrow H^0(\Omega^3) \longrightarrow H^1(\Omega^{2,\mathrm{cl}}) \longrightarrow H^1(\Omega^2) \longrightarrow 0 \quad (2.2)$$

We may interpret this as follows: given a holomorphic Courant algebroid E, the map to $H^1(\Omega^2)$ represents the isomorphism class of the extension (1.1), where we observe that the extension class, which a priori lies in $H^1(\Omega^1 \otimes \Omega^1)$, is forced to be skew by the orthogonal structure q. On the other hand, the inclusion $H^0(\Omega^3) \hookrightarrow H^1(\Omega^{2,\mathrm{cl}})$ can be seen from the fact that any Courant bracket $[\cdot,\cdot]$ may be modified, given a holomorphic $(3,0)$-form $H \in H^0(\Omega^3)$, as follows:

$$[e_1, e_2]_H := [e_1, e_2] + \iota(i_{\pi(e_1)} i_{\pi(e_2)} H).$$

In general, the $\partial\overline{\partial}$-lemma fails, and so the short exact sequence (2.2) may not hold. To further analyze this case, we use a Dolbeault resolution of $\Omega^{2,\mathrm{cl}}$, as follows. First resolve the sheaf using the local $\partial\overline{\partial}$-lemma:

$$0 \longrightarrow \Omega^{2,\mathrm{cl}} \longrightarrow \mathcal{Z}_\partial^{2,0} \xrightarrow{\overline{\partial}} \mathcal{Z}_\partial^{2,1} \xrightarrow{\overline{\partial}} \cdots,$$

where $\mathcal{Z}_\partial^{p,q}$ is the sheaf of smooth ∂-closed (p,q) forms. Then use the Dolbeault resolution for each $\mathcal{Z}_\partial^{p,q}$ given by ∂ to conclude that $H^1(\Omega^{2,\mathrm{cl}})$ is given by the first total cohomology of the double complex

$$\begin{array}{ccccc} \Omega^{4,0} & & & & \\ \uparrow{\scriptstyle\partial} & & & & \\ \Omega^{3,0} & \xrightarrow{\overline{\partial}} & \Omega^{3,1} & & \\ \uparrow{\scriptstyle\partial} & & \uparrow{\scriptstyle\partial} & & \\ \Omega^{2,0} & \xrightarrow{\overline{\partial}} & \Omega^{2,1} & \xrightarrow{\overline{\partial}} & \Omega^{2,2} \end{array} \tag{2.3}$$

Proposition 2.1. *Holomorphic Courant algebroids are classified up to isomorphism by*

$$H^1(\Omega^{2,\mathrm{cl}}) = \frac{(\Omega^{3,0}(M) \oplus \Omega^{2,1}(M)) \cap \ker d}{d(\Omega^{2,0}(M))}. \tag{2.4}$$

Example 2.2. Given a Dolbeault representative $T + H$ for a class in (2.4), with $T \in \Omega^{2,1}$ and $H \in \Omega^{3,0}$, we may construct a corresponding holomorphic Courant algebroid. Viewing T as a map $T : T_{1,0}M \longrightarrow T_{0,1}^*M \otimes T_{1,0}^*M$, define the following partial connection on sections of $E = T_{1,0}M \oplus T_{1,0}^*M$:

$$\overline{D} = \begin{pmatrix} \overline{\partial} & 0 \\ -T & \overline{\partial} \end{pmatrix} : \Gamma^\infty(E) \longrightarrow \Gamma^\infty(T_{0,1}^*M \otimes E),$$

where $\overline{\partial}$ are the usual holomorphic structures on the tangent and cotangent bundles. $\overline{D}$ squares to zero and defines a new holomorphic structure on the complex bundle E, which then becomes a holomorphic extension of $T_{1,0}$ by $T_{1,0}^*$. A section $X + \xi$ of E is $\overline{D}$-holomorphic if and only if $\overline{\partial}X = 0$ and $i_X T + \overline{\partial}\xi = 0$.

The symmetric pairing on E is the usual one obtained from the duality pairing, but the bracket is twisted as follows:

$$[X + \xi, Y + \eta] = [X, Y] + L_X\eta - i_Y d\xi + i_X i_Y h, \tag{2.5}$$

where $h := H + T$. Under the assumption that $d(T + H) = 0$, this bracket is well-defined on the sheaf of $\overline{D}$-holomorphic sections of E, and defines a holomorphic Courant algebroid, as required.

Example 2.3. Let X be the Hopf surface from Example 1.20. Its Hodge numbers $h^{p,q}$ all vanish except $h^{0,0} = 1$, $h^{0,1} = 1$, $h^{2,1} = 1$ and $h^{2,2} = 1$. Hence, the group (2.4) classifying holomorphic Courant algebroids is $H^1(\Omega^2) \cong \mathbb{C}$. The Courant algebroids in this 1-parameter family may be described explicitly using Example 2.2, but also holomorphically, as follows. The union of the pair of elliptic curves $E_1 = \{x_1 = 0\}$ and $E_2 = \{x_2 = 0\}$ is an anticanonical divisor, corresponding to the meromorphic section $B = c(x_1x_2)^{-1}dx_1 \wedge dx_2,\ c \in \mathbb{C}^*$. Glue $\mathbb{T}(X\backslash E_1)$ to $\mathbb{T}(X\backslash E_2)$ using the holomorphic closed 2-form B to obtain a Courant algebroid with modified extension class.

On any complex manifold, there is an injection of sheaves from the holomorphic closed 2-forms to the smooth real closed 2-forms,

$$\Omega^{2,\mathrm{cl}}_{\mathrm{hol}} \longrightarrow \Omega^{2,\mathrm{cl}}_{\infty}(\mathbb{R}), \qquad B^{2,0} \mapsto \frac{1}{2}(B^{2,0} + \overline{B^{2,0}}).$$

For this reason we have a map $H^1(\Omega^{2,\mathrm{cl}}_{hol}) \longrightarrow H^1(\Omega^{2,\mathrm{cl}}_{\infty})$; indeed, the underlying real vector bundle of any holomorphic Courant algebroid is itself a smooth real Courant algebroid.

Example 2.4. The real Courant algebroid on $X \cong S^3 \times S^1$ corresponding to the holomorphic Courant algebroid described in Example 2.3 is obtained by choosing a Dolbeault representative of the Čech cocycle $\frac{1}{2}(B+\overline{B})$. For $B = c(x_1x_2)^{-1}dx_1 \wedge dx_2,\ c \in \mathbb{C}$, we obtain a class in $H^3(X,\mathbb{R})$ which evaluates on the fundamental cycle of S^3 to $-4\pi^2\mathrm{Re}(c)$.

2.2. Holomorphic reduction. A generalized Kähler structure gives rise to a holomorphic Courant algebroid over each of the underlying complex manifolds X_-, X_+. To see this, our main tool will be holomorphic Courant reduction, developed in Appendix A, applied to the decomposition (1.21) induced by the generalized Kähler structure:

$$E \otimes \mathbb{C} = \ell_+ \oplus \ell_- \oplus \overline{\ell}_+ \oplus \overline{\ell}_-.$$

The bundles $\overline{\ell}_\pm$ satisfy $\pi(\overline{\ell}_\pm) = T^{0,1}_\pm M$, and hence are *liftings* of $T^{0,1}_\pm M$ to $E \otimes \mathbb{C}$ in the sense of Definition A.1. By Theorem A.5, we obtain the fact that $\mathcal{E}_\pm = \overline{\ell}^{\perp}_\pm / \overline{\ell}_\pm$ are holomorphic Courant algebroids over $X_\pm$.

Proposition 2.5. *The bundles $\overline{\ell}_\pm$ are liftings of $T_{0,1}X_\pm$ to $E \otimes \mathbb{C}$, hence define two reductions of $E \otimes \mathbb{C}$ to holomorphic Courant algebroids $\mathcal{E}_\pm$ over the complex manifolds $X_\pm$.*

$$\begin{array}{ccccc} & & (E \otimes \mathbb{C}, M) & & \\ & \overset{\overline{\ell}_-}{\swarrow} & & \overset{\overline{\ell}_+}{\searrow} & \\ (\mathcal{E}_-, X_-) & & & & (\mathcal{E}_+, X_+) \end{array} \tag{2.6}$$

Furthermore, the isomorphism class $[\mathcal{E}_\pm]$ is given by the (2, 1) *component (with respect to $I_\pm$) of the torsion of the generalized Kähler metric:*

$$[\mathcal{E}_\pm] = [2H^{(2,1)_\pm}] \in H^1(X_\pm; \Omega^{2,\mathrm{cl}}).$$

Proof. The existence of the reductions follows from Theorem A.5, as explained above. To compute the isomorphism classes, we write the lifting $\overline{\ell}_\pm$ explicitly using Eq. (1.23), namely $\ell_\pm = \mathrm{e}^{\mp i\omega_\pm}s(T_{1,0}X_\pm)$, and use the explicit form for the cocycle given in Remark A.3, yielding

$$[\mathcal{E}_\pm] = [H^{(2,1)_\pm} - \partial_\pm(\pm i\omega_\pm)].$$

Since $H^{(2,1)_\pm} = \mp i\partial_\pm\omega_\pm$ from (1.24), we obtain the required cocycle $2H^{(2,1)_\pm}$. □

Remark 2.6. For a more explicit description of $\mathcal{E}_\pm$, we may use the canonical splitting s given by the generalized Kähler metric to (smoothly) split the sequence

$$0 \longrightarrow T^*_{1,0}X_\pm \longrightarrow \mathcal{E}_\pm \longrightarrow T_{1,0}X_\pm \longrightarrow 0\,,$$

by defining the following map $s_\pm : T_{1,0}X_\pm \longrightarrow \mathcal{E}_\pm = \overline{\ell}_\pm^\perp/\overline{\ell}_\pm$:

$$s_\pm(X) := s(X) \mp gX = s(X) \pm i\omega_\pm X \mod \overline{\ell}_\pm,$$

for $X \in T_{1,0}X_\pm$. The holomorphic structure on $\mathcal{E}_\pm$ is then computed via (A.3), using the Courant bracket on $\mathbb{T}M$ given by the torsion 3-form H. The resulting Courant algebroid is $\mathcal{E}_\pm = T_{1,0}X_\pm \oplus T^*_{1,0}X_\pm$, with a modified holomorphic structure as in Example 2.2:

$$\overline{D}_\pm = \begin{pmatrix} \overline{\partial}_\pm & 0 \\ -2H^{(2,1)_\pm} & \overline{\partial}_\pm \end{pmatrix}.$$

The holomorphic Courant algebroids $(\mathcal{E}_\pm, X_\pm)$ are closely related, as they are both reductions of the same smooth Courant algebroid. Furthermore, the Lie algebroids $\overline{\ell}_-, \overline{\ell}_+$ are compatible in the sense that $\overline{\ell}_+ \oplus \overline{\ell}_- \subset E \otimes \mathbb{C}$ is a Dirac structure, hence a Lie algebroid. This configuration is well-known in the literature and is called a *matched pair* of Lie algebroids [38,40]. We now describe several consequences of these two compatible reductions.

Following the philosophy of symplectic reduction applied to Courant algebroids, we may also reduce Dirac structures from $E \otimes \mathbb{C}$ to $\mathcal{E}_\pm$, via Proposition A.6. Applying this to the Dirac structures given by the $\pm i$-eigenbundles of $\mathbb{J}_\pm$, we obtain holomorphic Dirac structures in $\mathcal{E}_\pm$.

Theorem 2.7. *Each of the holomorphic Courant algebroids $\mathcal{E}_\pm$ over the complex manifolds underlying a generalized Kähler manifold contains a pair of transverse holomorphic Dirac structures*

$$\mathcal{E}_\pm = \mathcal{A}_\pm \oplus \mathcal{B}_\pm,$$

where $\mathcal{A}_\pm$ are both the reduction of the $-i$-eigenbundle of $\mathbb{J}_+$ and $(\mathcal{B}_+, \mathcal{B}_-)$ are reductions of the $-i$ and $+i$ eigenbundles of $\mathbb{J}_-$, respectively.

Proof. Consider the reduction by $\overline{\ell}_-$. The Dirac structures in $E \otimes \mathbb{C}$ given by the $-i$-eigenbundle of $\mathbb{J}_+$ and the $+i$-eigenbundle of $\mathbb{J}_-$ are as follows:

$$\overline{L}_+ = \overline{\ell}_+ \oplus \overline{\ell}_-, \qquad\qquad L_- = \ell_+ \oplus \overline{\ell}_-.$$

Since they both contain $\overline{\ell}_-$ as an involutive subbundle, it follows that both Dirac structures have intersection with $\overline{\ell}_-^\perp$ of constant rank, and also that both $L_-, \overline{L}_+$ are $\overline{\ell}_-$-invariant. Hence by (A.4), they reduce to holomorphic Dirac structures $\mathcal{A}_-, \mathcal{B}_-$ in the holomorphic Courant algebroid $\mathcal{E}_-$. These are transverse simply because $\ell_+, \overline{\ell}_+$ have zero intersection. The same argument applies for the reduction by $\overline{\ell}_+$. □

Remark 2.8. Using the splittings $s_\pm$ from Remark 2.6, we can describe the Dirac structures explicitly as follows. For simplicity, we describe the Dirac structure $\mathcal{B}_+$ inside $\mathcal{E}_+ = \mathcal{A}_+ \oplus \mathcal{B}_+$. The Dirac structure $\mathcal{B}_+$ is obtained by reduction of $\overline{\ell}_+ \oplus \ell_-$, which has image in $\overline{\ell}_+^\perp/\overline{\ell}_+$ isomorphic to $\ell_- \cong T_{1,0}X_-$. Hence we give a map $T_{1,0}X_- \longrightarrow \mathcal{E}_+$ with image $\mathcal{B}_+$.

Let P_+ be the projection of a vector to $T_{1,0}X_+$, and let $\bar{P}_+$ be the complex conjugate projection. For any $X \in T_{1,0}X_-$, we have $X - gX \in \ell_-$, and therefore

$$\begin{aligned} X - gX &= (P_+X + \overline{P}_+X) - g(P_+X + \overline{P}_+X) \\ &= (P_+X - gP_+X) + (\overline{P}_+X - g\overline{P}_+X) \\ &= (P_+X - gP_+X) - 2g\overline{P}_+X \pmod{\overline{\ell}_+}, \end{aligned}$$

where the last two terms are in $s_+(T_{1,0}X_+)$ and $T^*_{1,0}X_+$, respectively. Hence the map

$$X \mapsto P_+X - 2g\overline{P}_+X$$

is an isomorphism $T_{1,0}X_- \longrightarrow \mathcal{B}_+$. In fact the same map gives an isomorphism $T_{0,1}X_- \longrightarrow \mathcal{A}_+$.

Remark 2.9. As complex bundles, $\mathcal{B}_\pm$ are isomorphic to $T_{(1,0)}X_\mp$. In other words, $\mathcal{B}_+$, a holomorphic Dirac structure on X_+, is isomorphic as a smooth bundle to the holomorphic tangent bundle of the opposite complex manifold X_-, and vice versa for $\mathcal{B}_-$. The way in which the holomorphic tangent bundle of X_- acquires a holomorphic structure with respect to X_+ seems particularly relevant to the study of so-called heterotic compactifications with (2, 0) supersymmetry [34,47], where only one of the complex structures $I_\pm$ is present, but there is an auxiliary holomorphic bundle which appears to play a similar role to $\mathcal{B}_\pm$.

The presence of transverse Dirac structures in each of $\mathcal{E}_\pm$ immediately implies, by (1.6) and the surrounding discussion, that $X_\pm$ inherit holomorphic Poisson structures. We now describe these explicitly, and verify that they coincide with the holomorphic Poisson structures discovered by Hitchin [29].

Proposition 2.10. *By forming the Baer sum $\mathcal{A}_\pm^\top \boxtimes \mathcal{B}_\pm$, the transverse Dirac structures $\mathcal{A}_\pm, \mathcal{B}_\pm$ give rise to holomorphic Poisson structures $\sigma_\pm$ on the complex manifolds $X_\pm$, both of which have real part*

$$\mathrm{Re}(\sigma_\pm) = \frac{1}{8}g^{-1}[I_+^*, I_-^*].$$

Proof. Following (1.7), we compute $\sigma_\pm$ explicitly using the decomposition $\overline{\ell}_\pm^\perp = \ell_\mp \oplus \overline{\ell}_+ \oplus \overline{\ell}_-$ and the canonical splitting of E given by the generalized Kähler metric. Let $P_\pm = \frac{1}{2}(1 - iI_\pm)$ be the projection of a vector or covector to its $(1,0)_\pm$ part, and let $\overline{P}_\pm$ be its complex conjugate. Then $\sigma_\pm$ applied to $\xi \in T^*X_\pm \otimes \mathbb{C}$ is given by taking the component α of $P_\pm\xi \in \overline{\ell}_\pm^\perp$ along $\ell_\mp$, and then projecting it to $(TX_\pm \otimes \mathbb{C})/T_{0,1}X_\pm$. Computing α, we obtain

$$(\overline{P}_\mp P_\pm\xi) \mp g^{-1}(\overline{P}_\mp P_\pm\xi),$$

and projecting α we obtain $\mp P_\pm g^{-1}\bar{P}_\mp P_\pm\xi$, so that our expression for $\sigma_\pm$ is

$$\begin{aligned} \sigma_\pm &= \mp g^{-1}\overline{P}_\pm\overline{P}_\mp P_\pm \\ &= \mp\frac{1}{8}g^{-1}(1 + iI_\pm^*)(1 + iI_\mp^*)(1 - iI_\pm^*) \\ &= \frac{1}{8}g^{-1}([I_+^*, I_-^*] + iI_\pm^*[I_+^*, I_-^*]). \end{aligned} \tag{2.7}$$

□

Example 2.11. To give a description of the Dirac structures $(\mathcal{A}_-, \mathcal{B}_-)$ for the generalized Kähler structure on the Hopf surface from Example 1.21, we compute the isomorphism $T_{0,1}X_+ \longrightarrow \mathcal{A}_-$ as in Remark 2.8, yielding $X \mapsto P_-X + 2g\overline{P_-X}$, and apply it to the basis of $(0,1)_+$ vectors given by $R^{-2}g^{-1}(\bar{x}_1 dx_1 + x_2 d\bar{x}_2)$ and $R^{-2}g^{-1}(\bar{x}_1 dx_2 - x_2 d\bar{x}_1)$, obtaining the following basis of holomorphic sections for $\mathcal{A}_-$:

$$x_2\frac{\partial}{\partial x_2} + \frac{1}{2\pi R^2}\bar{x}_1 dx_1, \qquad -x_2\frac{\partial}{\partial x_1} + \frac{1}{2\pi R^2}\bar{x}_1 dx_2.$$

The same prescription produces a basis for $\mathcal{B}_-$:

$$x_1\frac{\partial}{\partial x_1} + \frac{1}{2\pi R^2}\bar{x}_2 dx_2, \qquad x_1\frac{\partial}{\partial x_2} - \frac{1}{2\pi R^2}\bar{x}_2 dx_1.$$

We see from this that the anchor map for $\mathcal{A}_-$ is an isomorphism except along the curve $\{x_2 = 0\}$ where it has rank zero, whereas the anchor map for $\mathcal{B}_-$ drops rank along $\{x_1 = 0\}$. Computing the Poisson tensor σ_- using (2.7) yields

$$\sigma_- = -x_1 x_2 \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2},$$

which is an anticanonical section vanishing on the union of the degeneration loci of $\mathcal{A}_-$ and $\mathcal{B}_-$.

The fact that the $\pm i$ eigenbundles of $\mathbb{J}_\pm$ descend to transverse holomorphic Dirac structures provides a great deal of information concerning the classical geometry that they determine on the base manifold. Just as in Sect. 1.1, where we discussed the transverse singular foliations induced on a manifold by transverse Dirac structures, we have a similar result here.

Proposition 2.12. *The transverse holomorphic Dirac structures $\mathcal{A}_\pm, \mathcal{B}_\pm$ in $\mathcal{E}_\pm$ induce transverse holomorphic singular foliations $\mathcal{F}_\pm, \mathcal{G}_\pm$ on $X_\pm$. The intersection of a leaf of $\mathcal{F}_\pm$ with a leaf of $\mathcal{G}_\pm$ is a (possibly disconnected) symplectic leaf for the holomorphic Poisson structure $\sigma_\pm$. Furthermore, the singular foliations $\mathcal{F}_\pm, \mathcal{G}_\pm$ coincide with the foliations induced by the generalized complex structures $\mathbb{J}_+, \mathbb{J}_-$, respectively.*

Proof. The behaviour of the holomorphic Dirac structures is precisely as in the real case, discussed in Sect. 1.1. To see why the holomorphic foliations coincide with the generalized complex foliations, we may appeal to the reduction procedure for Dirac structures, which makes it evident.

Alternatively, observe that in order to extract a foliation from a holomorphic Lie algebroid $\mathcal{A}$ over X, one possible way to proceed is to first represent the holomorphic Lie algebroid as a smooth Lie algebroid with compatible holomorphic structure, by forming the associated complex Lie algebroid $A = \mathcal{A} \oplus T_{0,1}X$ as in [33]. Then take the fiber product over $TX \otimes \mathbb{C}$ with the complex conjugate $\overline{A} = \overline{\mathcal{A}} \oplus T_{1,0}X$, to obtain a real Lie algebroid, defining a foliation of X.

Applying this to the Lie algebroid $\mathcal{A}_\pm$ over $X_\pm$, we see immediately that the associated complex Lie algebroid $\mathcal{A}_\pm \oplus T_{0,1}X_\pm$ is precisely $\overline{\ell}_+ \oplus \overline{\ell}_- = \overline{L}_+$, the $-i$-eigenbundle of $\mathbb{J}_+$. Furthermore, the fiber product construction yields

$$\mathcal{A}'_\pm \otimes_{TX_\pm \otimes \mathbb{C}} \overline{\mathcal{A}'_\pm} = \overline{L_+} \otimes_{TX_\pm \otimes \mathbb{C}} L_+,$$

which by (1.10) is the Lie algebroid corresponding to the real Poisson structure associated to $\mathbb{J}_+$. The same argument applies to $\mathcal{B}_\pm$, relating its holomorphic foliation to that determined by $\mathbb{J}_-$. □

Remark 2.13. According to the above proposition, the generalized foliation induced by a generalized complex structure $\mathbb{J}_\pm$ in a generalized Kähler pair is holomorphic with respect to I_+ and I_-. The relationship between the symplectic structure of the leaves and the induced complex structures from $I_\pm$ may be understood by applying the theory of generalized Kähler reduction, as follows.

Let $S \subset M$ be a submanifold and $K = N^*S$ its conormal bundle. We saw in (1.8) that a Courant algebroid E on M may be pulled back to S to yield a Courant algebroid $E_S = K^\perp/K$ over S. If a generalized complex structure $\mathbb{J}$ on E satisfies $\mathbb{J}K \subset K$, then it induces a generalized complex structure $\mathbb{J}_{\mathrm{red}}$ on the reduced Courant algebroid E_S. If S is a leaf of the real Poisson structure Q associated to $\mathbb{J}$, it follows that $\mathbb{J}K \subset K$ and that $\mathbb{J}_{\mathrm{red}}$ is of symplectic type, reproducing the symplectic structure derived from Q.

In [8,9], it is shown that if $(\mathbb{J}_+, \mathbb{J}_-)$ is a generalized Kähler structure for which $\mathbb{J}_+K = K$ as above, then the entire generalized Kähler structure reduces to E_S, with $(\mathbb{J}_+)_{\mathrm{red}}$ of symplectic type. In particular, we obtain a bi-Hermitian structure on S. We may then perform a second generalized Kähler reduction, from S to a symplectic leaf of $(\mathbb{J}_-)_{\mathrm{red}}$, whereupon we obtain a generalized Kähler structure where both generalized complex structures are of symplectic type.

2.3. Sheaves of differential graded Lie algebras. A Dirac structure $\mathcal{A} \subset \mathcal{E}$ is, in particular, a Lie algebroid, and has a Lie algebroid de Rham complex $(\Omega^\bullet_{\mathcal{A}}, d_{\mathcal{A}})$. If $(\mathcal{A}, \mathcal{B})$ is a pair of transverse Dirac structures, then as was observed in [37], the de Rham complex inherits further structure. It is shown there that if we make the identification $\mathcal{B} = \mathcal{A}^*$ using the symmetric pairing on $\mathcal{E}$, then the Lie bracket on $\mathcal{B}$ extends to a differential graded Lie algebra structure on $\Omega^\bullet_{\mathcal{A}}$, so that

$$(\Omega^\bullet_{\mathcal{A}}, d_{\mathcal{A}}, [\cdot,\cdot]_{\mathcal{B}})$$

is a sheaf of differential graded Lie algebras (the degree is shifted so that $\Omega^k_{\mathcal{A}}$ has degree $k-1$).

Given a differential graded Lie algebra as above, there is a natural question which arises: what is the object whose deformation theory it controls? In [37], the above differential graded Lie algebra was explored in the smooth category, in which case there is a direct interpretation in terms of deformations of Dirac structures. A deformation of the Dirac structure $\mathcal{A}$ inside $\mathcal{E} = \mathcal{A} \oplus \mathcal{B}$ may be described as the graph of a section $\epsilon \in \Omega^2_{\mathcal{A}}(M)$, viewed as a map $\epsilon : \mathcal{A} \longrightarrow \mathcal{B} = \mathcal{A}^*$. It is shown in [37] that the involutivity of this graph is equivalent to the Maurer–Cartan equation:

$$d_{\mathcal{A}}\epsilon + \frac{1}{2}[\epsilon, \epsilon]_{\mathcal{B}} = 0. \tag{2.8}$$

This leads, assuming that $(\Omega^\bullet_{\mathcal{A}}, d_{\mathcal{A}})$ is an elliptic complex and M is compact, to a finite-dimensional moduli space of deformations of $\mathcal{A}$ in $\mathcal{E}$, presented as the zero set of an obstruction map $H^2_{d_{\mathcal{A}}} \longrightarrow H^3_{d_{\mathcal{A}}}$.

The deformation theory governed by a sheaf of differential graded Lie algebras in the holomorphic category is much more subtle, for the reason that the objects being deformed are not required to be given by global sections of the sheaf (of which there may be none). The objects are considered to be "derived" in the sense that the Maurer–Cartan equation (2.8) is not applied to global sections in $\Omega^2_{\mathcal{A}}(X)$ but rather to the global sections in total degree 2 of a resolution $\mathcal{I}^{\bullet\bullet}$ of the complex $\Omega^\bullet_{\mathcal{A}}$. Note that the structure of the resolution $\mathcal{I}^{\bullet\bullet}$ may not, in general, be that of a differential graded Lie algebra, but only one up to homotopy, so one must interpret the Maurer–Cartan equation appropriately. In any case, the moduli space is then given by an obstruction map between the derived global sections of the differential complex $(\Omega^\bullet_{\mathcal{A}}, d_{\mathcal{A}})$, namely the hypercohomology groups. In short, we expect a moduli space described by an obstruction map

$$\mathbb{H}^2(\Omega^\bullet_{\mathcal{A}}, d_{\mathcal{A}}) \longrightarrow \mathbb{H}^3(\Omega^\bullet_{\mathcal{A}}, d_{\mathcal{A}}).$$

General results concerning such deformation theories can be found, for example, in [26,48], and a case relevant to generalized geometry has been investigated in [12].

We wish simply to observe that in our case, since the holomorphic Dirac structures $(\mathcal{A}_\pm, \mathcal{B}_\pm)$ are obtained by reduction from smooth Dirac structures in $E \otimes \mathbb{C}$, their de Rham complexes are equipped with canonical resolutions by fine sheaves, which are themselves differential graded Lie algebras controlling a known deformation problem. We conclude with the main result of this section, which may be viewed as a holomorphic description for the deformation theory of generalized complex structures, under the assumption of the generalized Kähler condition.

Proposition 2.14. *The derived deformation complex defined by the sheaf of holomorphic differential graded algebras* $(\Omega^\bullet_{\mathcal{A}_+}, d_{\mathcal{A}_+}, [\cdot,\cdot]_{\mathcal{B}_+})$ *on the complex manifold* X_+ *is canonically isomorphic to that defined by* $(\Omega^\bullet_{\mathcal{A}_-}, d_{\mathcal{A}_-}, [\cdot,\cdot]_{\mathcal{B}_-})$ *on the complex manifold* X_-*: they both yield the deformation complex of the generalized complex structure* $\mathbb{J}_+$.

Similarly, the sheaves of differential graded Lie algebras $(\Omega^\bullet_{\mathcal{B}_\pm}, d_{\mathcal{B}_\pm}, [\cdot,\cdot]_{\mathcal{A}_\pm})$ *have derived deformation complexes which are canonically complex conjugate to each other, and are naturally isomorphic to the deformation complex of the generalized complex structure* $\mathbb{J}_-$.

Proof. Consider the $-i$ eigenbundle of $\mathbb{J}_+$, given by $\overline{L}_+ = \overline{\ell}_+ \oplus \overline{\ell}_-$. Because $\overline{L}_+$ decomposes into the involutive Lie sub-algebroids $\overline{\ell}_\pm$, its de Rham complex is the total complex of a double complex:

$$\Omega^k_{\overline{L}_+} = \bigoplus_{p+q=k} \mathcal{O}(\wedge^p \overline{\ell}_-^* \otimes \wedge^q \overline{\ell}_+^*), \qquad d_{\overline{L}_+} = d_{\overline{\ell}_-} + d_{\overline{\ell}_+}. \tag{2.9}$$

Identifying $\ell_\pm^*$ with $\overline{\ell}_\pm$ using the symmetric pairing on E, the above double complex inherits a Lie bracket from the Lie algebroid $L_+ = \ell_- \oplus \ell_+$. Furthermore, since $(L_+, \overline{L}_+)$ forms a Lie bialgebroid, we obtain that the Lie bracket on (2.9) is compatible with the bi-grading and the differential. Finally, recall that $\overline{\ell}_\pm$ is isomorphic to $T_{(0,1)}X_\pm$. As a result, we may view the differential $\mathbb{Z} \times \mathbb{Z}$-graded Lie algebra (2.9) in two ways:

i) Horizontally, using the differential $d_{\overline{\ell}_-}$, the complex is a Dolbeault resolution, over the complex manifold X_-, of the de Rham complex of the holomorphic Lie algebroid $\mathcal{A}_-$. The inclusion of $\Omega^\bullet_{\mathcal{A}_-}$ in the double complex is also a homomorphism of Lie algebras.
ii) Vertically, using the differential $d_{\overline{\ell}_+}$, the complex is a Dolbeault resolution, over X_+, of the de Rham complex of $\mathcal{A}_+$. Also, the inclusion of $\Omega^\bullet_{\mathcal{A}_+}$ is a homomorphism of Lie algebras.

On the other hand, the total complex of this double complex has already been interpreted; as we saw in Sect. 1.1, the differential graded Lie algebra $(\Omega^\bullet_{\overline{L}_+}, d_{\overline{L}_+}, [\cdot,\cdot]_{L_+})$ controls the deformation theory of the generalized complex structure $\mathbb{J}_+$. The statement for $(\Omega^\bullet_{\mathcal{B}_\pm}, d_{\mathcal{B}_\pm}, [\cdot,\cdot]_{\mathcal{A}_\pm})$ is shown in the same way, using instead the $\pm i$-eigenbundles of $\mathbb{J}_-$. □

In particular, the above result implies the following fact, striking from the point of view of the complex manifolds $X_\pm$, which are not related in any obvious holomorphic fashion:

Corollary 2.15. *We have the following canonical isomorphisms of hypercohomology for the de Rham complexes of the holomorphic Dirac structures* $(\mathcal{A}_-, \mathcal{B}_-)$ *on* X_- *and* $(\mathcal{A}_+, \mathcal{B}_+)$ *on* X_+*:*

$$\mathbb{H}^k(X_-, \Omega^\bullet_{\mathcal{A}_-}) = \mathbb{H}^k(X_+, \Omega^\bullet_{\mathcal{A}_+});$$
$$\mathbb{H}^k(X_-, \Omega^\bullet_{\mathcal{B}_-}) = \overline{\mathbb{H}^k(X_+, \Omega^\bullet_{\mathcal{B}_+})}.$$

2.4. Morita equivalence. In the previous section, we saw that the pair $(\mathcal{A}_+, \mathcal{B}_+)$ of transverse holomorphic Dirac structures on the complex manifold X_+ is closely related to its counterpart $(\mathcal{A}_-, \mathcal{B}_-)$ on X_-, in that the Dirac structures $\mathcal{A}_\pm$ have identical derived deformation theory and hypercohomology groups, and similarly for $\mathcal{B}_\pm$. The purpose of this section is to describe the relationship between $(X_+, \mathcal{A}_+, \mathcal{B}_+)$ and $(X_-, \mathcal{A}_-, \mathcal{B}_-)$ as a Morita equivalence. Morita equivalence for Lie algebroids in the smooth category is well-studied in Poisson geometry [14,16,50] and the version we develop here is a special case, with additional refinements made possible by the complex structures which are present. We use the result from [33] that a holomorphic Lie algebroid $\mathcal{L}$ on X may be described equivalently by a complex Lie algebroid structure on $L = \mathcal{L} \oplus T_{0,1}X$, compatible with the given holomorphic data.

Definition 2.16. *Let* $\varphi_\pm : M \longrightarrow X_\pm$ *be diffeomorphisms from a manifold* M *to two complex manifolds* $X_\pm$*, and let* $\mathcal{L}_\pm$ *be holomorphic Lie algebroids on* $X_\pm$*. Then* $\mathcal{L}_+$ *is Morita equivalent to* $\mathcal{L}_-$ *when there is an isomorphism* ψ *between the associated complex Lie algebroids* $L_\pm := \mathcal{L}_\pm \oplus T_{0,1}X_\pm$*, i.e.:*

$$\begin{array}{ccc} \varphi_+^* L_+ & \xrightarrow{\ \psi\ } & \varphi_-^* L_- \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

Similarly, $\mathcal{L}_+$ *is Morita conjugate to* $\mathcal{L}_-$ *when there is an isomorphism of complex Lie algebroids from* L_+ *to* $\overline{L_-}$*.*

Proposition 2.17. *Let* $(\mathcal{A}_\pm, \mathcal{B}_\pm)$ *be the transverse holomorphic Dirac structures on the complex manifolds* $X_\pm$ *participating in a generalized Kähler structure. Then* $\mathcal{A}_+$ *is Morita equivalent to* $\mathcal{A}_-$*, and* $\mathcal{B}_+$ *is Morita conjugate to* $\mathcal{B}_-$*.*

Proof. This is an immediate consequence of the canonical isomorphisms of complex Lie algebroids

$$A_+ = \mathcal{A}_+ \oplus T_{0,1}X_+ = \overline{\ell}_- \oplus \overline{\ell}_+ = \mathcal{A}_- \oplus T_{0,1}X_- = A_-$$
$$B_+ = \mathcal{B}_+ \oplus T_{0,1}X_+ = \ell_- \oplus \overline{\ell}_+ = \overline{\mathcal{B}_-} \oplus T_{1,0}X_- = \overline{B}_-.$$

□

Just as in [16], the Morita equivalence between $\mathcal{A}_+, \mathcal{A}_-$ induces an equivalence between their $\mathbb{C}$-linear categories of modules. Similarly the Morita conjugacy between $\mathcal{B}_+, \mathcal{B}_-$ implies a $\mathbb{C}$-antilinear equivalence of their module categories. Since the Morita equivalence is an isomorphism at the level of complex Lie algebroids over M, we can also strengthen this statement to an equivalence of the DG categories of cohesive modules [5], i.e. representations up to homotopy [1]. We only remark here that these modules have a generalized complex interpretation, since $\overline{L}_\pm = \overline{\ell}_+ \oplus \overline{\ell}_-$ is the $-i$-eigenbundle of $\mathbb{J}_+$, whose modules are, by definition, generalized holomorphic bundles [25].

Corollary 2.18. *The categories of holomorphic $\mathcal{A}_\pm$-modules are equivalent to each other and to the category of generalized holomorphic bundles for $\mathbb{J}_+$. Similarly, the category of holomorphic $\mathcal{B}_+$-modules is equivalent to the category of generalized holomorphic bundles for $\mathbb{J}_-$, and $\mathbb{C}$-antilinearly equivalent to the category of modules for $\mathcal{B}_-$.*

A special case occurs when $\mathbb{J}_-$ is of symplectic type; in this case $\mathcal{B}_\pm$ are isomorphic as holomorphic Lie algebroids with $T_{1,0}X_\pm$. But $T_{1,0}X_+$ is a holomorphic Lie algebroid which is actually Morita conjugate to itself, via the complex conjugation map

$$T_{1,0}X_+ \oplus T_{0,1}X_+ \xrightarrow{c.c.} T_{1,0}X_+ \oplus T_{0,1}X_+.$$

Hence, composing this with the Morita conjugacy $\mathcal{B}_+ \longrightarrow \mathcal{B}_-$, we obtain that $\mathcal{B}_+$ is Morita equivalent to $\mathcal{B}_-$. This is significant because we then have a Morita equivalence between the fiber product of the Lie algebroids over $T_{1,0}X_\pm$:

$$\mathcal{A}_+ \oplus_{T_{1,0}X_+} \mathcal{B}_+ \longrightarrow \mathcal{A}_- \oplus_{T_{1,0}X_-} \mathcal{B}_-.$$

But by Proposition 2.10, these fiber products are the holomorphic Lie algebroids corresponding to the holomorphic Poisson structures $\sigma_\pm$ on $X_\pm$. Hence we obtain a Morita equivalence between holomorphic Poisson structures, generalizing the result in [24] on Morita equivalence for a specific construction of generalized Kähler structures.

Corollary 2.19. *If either $\mathbb{J}_+$ or $\mathbb{J}_-$ is of symplectic type, then the holomorphic Poisson structures $\sigma_\pm$ on $X_\pm$ are Morita equivalent.*

2.5. Prequantization and holomorphic gerbes. Geometric quantization of symplectic manifolds is perhaps best understood in the setting of Kähler geometry. A symplectic manifold (M, ω) is said to be prequantizable when $[\omega]/2\pi \in H^2(M, \mathbb{R})$ has integral periods, i.e. is in the image of the natural map $H^2(M, \mathbb{Z}) \longrightarrow H^2(M, \mathbb{R})$. A prequantization of such an integral symplectic form is a Hermitian complex line bundle (L, h) equipped with a unitary connection ∇ such that $F(\nabla) = i\omega$. The presence of a complex structure I compatible with ω, sometimes called a complex polarization, then implies that $\nabla^{0,1}$ defines a holomorphic structure on the line bundle L, which is used to proceed with the geometric quantization procedure. In this sense, we view a Hermitian holomorphic line bundle over a complex manifold (M, I, L, h) as a prequantization of the Kähler structure (M, I, ω). In this section, we seek an analogous result for generalized Kähler manifolds.

Our first task is to prequantize the underlying Courant algebroid E. For this to be possible, we need the quantization condition that $[E]/2\pi \in H^3(M, \mathbb{R})$ has integral periods, and choose a Hermitian gerbe (G, h) with unitary 0-connection ∇ such that the associated Courant algebroid E_∇ (via Corollary B.6) satisfies $[E_\nabla] = [E]$. This is always possible since the map (B.3) is surjective onto classes vanishing in $H^3(M, \mathbb{R}/\mathbb{Z})$.

If E_∇ carries a generalized complex structure $\mathbb{J}$, it immediately obtains Dirac structures $L, \overline{L}$ given by $\ker(\mathbb{J} \mp i)$, and by Theorem B.7, this induces flat connections on G over these Lie algebroids. By analogy with vector bundles, we say that a gerbe with a flat $\overline{L}$-connection is a generalized holomorphic gerbe.

Proposition 2.20. *Let $\mathbb{J}$ be a generalized complex structure on E_∇, where ∇ is a unitary 0-connection on the Hermitian gerbe (G, h). Then G inherits a generalized holomorphic structure. Furthermore, it has a flat Poisson connection for the underlying real Poisson structure.*

Proof. G inherits a flat $\overline{L}$-connection by Theorem B.7. To show that it has a flat Poisson connection, we note that the trivial gerbe has a canonical flat L-connection, and by tensoring with G we obtain a flat connection on G, for the fiber product of $\overline{L}$ with L, which by (1.10) is the Lie algebroid of the Poisson structure Q underlying $\mathbb{J}$, as required. □

Applying the above result to each generalized complex structure separately, a generalized Kähler structure $(\mathbb{J}_+, \mathbb{J}_-)$ on E_∇ induces flat $\overline{L}_\pm$-connections on G, rendering it generalized holomorphic with respect to both $\mathbb{J}_\pm$.

Corollary 2.21. *Let $(\mathbb{J}_+, \mathbb{J}_-)$ be a generalized Kähler structure on E_∇, which is as above. Then the gerbe G inherits generalized holomorphic structures over $\mathbb{J}_+$ and $\mathbb{J}_-$. In particular, it obtains flat Poisson connections over the real Poisson structures $Q_\pm$ underlying $\mathbb{J}_\pm$.*

We may interpret the above generalized holomorphic structures in terms of the underlying bi-Hermitian geometry, as follows.

Proposition 2.22. *Let $(G, h, \nabla, \mathbb{J}_+, \mathbb{J}_-)$ be as above. Then G inherits holomorphic structures with respect to the underlying complex manifolds $X_\pm$, defining holomorphic gerbes $\mathcal{G}_\pm$. Furthermore, $\mathcal{G}_\pm$ inherit holomorphic 0-connections $\partial_\pm$, whose associated holomorphic Courant algebroids $\mathcal{E}_\pm$ are given in Proposition 2.5.*

Proof. The commuting of $\mathbb{J}_\pm$ induces the decomposition (1.19), and since $\overline{\ell}_\pm$ are liftings of $T_{0,1}X_\pm$, Theorem B.8 implies that the gerbe G inherits the structure of a holomorphic gerbe with holomorphic 0-connection over $X_\pm$, with associated holomorphic Courant algebroid $\mathcal{E}_\pm$ given by the reduction of Courant algebroids in Proposition 2.5. □

Remark 2.23. The fact that the gerbe G inherits a pair of holomorphic structures with respect to $X_\pm$ was proposed in the papers [31,36,51], which contain more insights concerning the prequantization of generalized Kähler geometry than are formalized here.

By the holomorphic reduction procedure developed in Sect. 2.2, we saw that the Dirac structures $\overline{L}_\pm$ reduce to the pair of holomorphic Dirac structures $\mathcal{A}_\pm, \mathcal{B}_\pm \subset \mathcal{E}_\pm$. Theorem B.7 then immediately yields the following result, which may be interpreted as establishing a relationship between the holomorphic gerbes $\mathcal{G}_\pm$ over $X_\pm$ deriving from the fact that $X_\pm$ participate in a generalized Kähler structure.

Theorem 2.24. *Let $(\mathcal{G}_\pm, \partial_\pm)$ be the holomorphic gerbe with 0-connection obtained as above from a generalized Kähler structure with prequantized Courant algebroid. Then $\mathcal{G}_\pm$ has flat connections over the holomorphic Lie algebroids $\mathcal{A}_\pm, \mathcal{B}_\pm$, and consequently has a flat Poisson connection with respect to the holomorphic Poisson structure $\sigma_\pm$.*

Proof. The decomposition $\mathcal{E}_\pm = \mathcal{A}_\pm \oplus \mathcal{B}_\pm$ obtained in Theorem 2.7 implies, by Theorem B.7, that $\mathcal{G}_\pm$ obtains flat connections over $\mathcal{A}_\pm$ and $\mathcal{B}_\pm$. By Proposition 2.10, the Baer sum $\mathcal{A}_\pm^\top \boxtimes \mathcal{B}_\pm$ yields the Lie algebroid of the holomorphic Poisson structure $\sigma_\pm$, and since the Baer sum coincides with the fiber product of Lie algebroids (Proposition 1.5), we obtain a flat Poisson connection on $\mathcal{G}_\pm$ with respect to $\sigma_\pm$. □

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A Holomorphic Courant Reduction

Definition A.1. *Let E be a complex Courant algebroid over a complex manifold X. A* lifting *of $T_{0,1}X$ to E is an isotropic, involutive subbundle $D \subset E$ mapping isomorphically to $T_{0,1}X$ under $\pi : E \longrightarrow TX \otimes \mathbb{C}$.*

The existence of a lifting for $T_{0,1}X$ as above will be controlled by an obstruction map which we now describe. Consider the short exact sequence of vertical complexes:

$$\begin{array}{ccccccc} & & & & \Omega^1_X & \xrightarrow{\partial} & \Omega^{2,\mathrm{cl}}_X \longrightarrow 0 \\ & & & & \uparrow{\scriptstyle\partial} & & \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathcal{O}_X & & \end{array}$$

This gives the following excerpt from the long exact sequence:

$$H^1(\Omega^{2,\mathrm{cl}}_X) \xrightarrow{\ \epsilon\ } H^3(X, \mathbb{C}) \xrightarrow{\ \gamma\ } \mathbb{H}^3(\mathcal{O}_X \xrightarrow{\partial} \Omega^1_X) \tag{A.1}$$

Lemma A.2. *Let E be a complex Courant algebroid over the complex manifold X. There exists a lifting of $T_{0,1}$ to E if and only if $\gamma([E]) = 0$ in $\mathbb{H}^3(\mathcal{O}_X \xrightarrow{\partial} \Omega^1_X)$.*

Proof. Choose an isotropic splitting $s : TX \longrightarrow E$, which determines a 3-form H as in Eq. (1.14), so that E is isomorphic as a Courant algebroid to $\mathbb{T}X \otimes \mathbb{C}$ equipped with the Courant bracket twisted by H as in Eq. (2.5); indeed $[E] = [H] \in H^3(M, \mathbb{C})$. Then a general isotropic lifting of $T^{0,1}X$ is given by

$$D = \{X + i_X\theta \ : \ X \in T^{0,1}X, \ \ \theta \in \Omega^{1,1}(X) \oplus \Omega^{0,2}(X)\},$$

and D is involutive if and only if $(d\theta - H)^{(1,2)+(0,3)} = 0$, or in other words

$$H^{1,2} + H^{0,3} = \overline{\partial}\theta^{1,1} + d\theta^{0,2}. \tag{A.2}$$

Using the Dolbeault resolution of $\mathcal{O}_X \xrightarrow{\partial} \Omega^1_X$, we conclude that θ exists if and only if $\gamma([H]) = 0$. □

Remark A.3. A solution to Eq. (A.2) defines a cocycle $H^{3,0} + H^{2,1} - \partial\theta^{1,1} \in \mathcal{Z}^1(\Omega^{2,\mathrm{cl}}_X)$ [using the resolution (2.3)], since

$$d(H^{3,0} + H^{2,1} - \partial\theta^{1,1}) = \overline{\partial} H^{2,1} + \partial\overline{\partial}\theta^{1,1} = \overline{\partial} H^{2,1} + \partial(H^{1,2} - \partial\theta^{0,2}) = 0.$$

Furthermore, we may change the isotropic splitting s in the proof above by a global smooth 2-form B, which sends $H \mapsto H + dB$ and modifies the lifting via $\theta \mapsto \theta + B^{1,1} + B^{0,2}$. As a result, the cocycle condition (A.2) holds independently of the choices made. In this way, a lifting of $T_{0,1}X$ to E naturally induces a lifting of $[E] \in H^3(M, \mathbb{C})$ to $H^1(\Omega^{2,\mathrm{cl}}_X)$ in the exact sequence (A.1).

Remark A.4. The map ϵ in (A.1) is lifted to a natural operation on Courant algebroids in [21], where it is shown that any holomorphic Courant algebroid $\mathcal{E}$ induces a smooth complex Courant algebroid structure on $\mathcal{E} \oplus (T_{0,1}M \oplus T^*_{0,1}M)$, called the companion matched pair of $\mathcal{E}$.

We see from (A.1) that if $\gamma([H]) = 0$, then $[H]$ is in the image of a map from $H^1(\Omega^{2,\mathrm{cl}}_X)$, which classifies holomorphic Courant algebroids. Indeed, Lemma A.2 and the above remark imply that a complex Courant algebroid with a lifting of $T_{0,1}X$ gives rise to a natural holomorphic Courant algebroid, as we now show.

Theorem A.5. *Let X be a complex manifold. A lifting D of $T_{0,1}X$ to a complex Courant algebroid E gives rise to a natural holomorphic Courant algebroid E_D on X.*

Proof. As in the proof of Lemma A.2, an isotropic splitting $s : TX \longrightarrow E$ gives rise to a 3-form h_s and an isomorphism of Courant algebroids $s_* : (E, [\cdot,\cdot]) \longrightarrow (\mathbb{T}X, [\cdot,\cdot]_{h_s})$. Using this splitting, the lifting $D \subset E$ of $T_{0,1}X$ is given by a 2-form $\theta_s \in \Omega^{1,1}(X) \oplus \Omega^{0,2}(X)$. By the above remark, we also see that the 3-form $h_s^{3,0} + h_s^{2,1} - \partial\theta_s^{1,1}$ is a cocycle and therefore defines a holomorphic Courant algebroid on $\mathbb{T}_{1,0}X = T_{1,0}X \oplus T^*_{1,0}X$ via the construction in Example 2.2, taking the cocycle (T, H) in that example to be $T_s = h_s^{2,1} - \partial\theta_s^{1,1}$ and $H_s = h_s^{3,0}$. Finally, the equivariance described in the above remark proves that the induced holomorphic Courant algebroid structure on $E_D = \mathbb{T}_{1,0}X$ varies functorially with the choices. □

To obtain the holomorphic Courant algebroid described above in a more direct way, we use the reduction procedure for Courant algebroids described in [8]. The lifting $D \subset E$ of $T_{0,1}X$ defines an "extended action" of $T_{0,1}X$ on E, and we perform a generalization of the symplectic quotient construction for the Courant algebroid E, as follows.

The reduction of E by D is given as an orthogonal bundle by $E_D = D^\perp/D$, where $D^\perp$ is the orthogonal complement of D with respect to the symmetric pairing on E. Note that since D is a lifting of $T_{0,1}X$, the kernel of $\pi|_{D^\perp}$ is $D^\perp \cap (T^* \otimes \mathbb{C}) = T^*_{1,0}X$, and therefore $E_D = D^\perp/D$ is an extension of the form

$$0 \longrightarrow T^*_{1,0}X \longrightarrow E_D \longrightarrow T_{1,0}X \longrightarrow 0\,.$$

The holomorphic structure on E_D is a natural consequence of the general fact that the bundle $D^\perp/D$ inherits a flat connection over the Lie algebroid D: given $s \in \Gamma^\infty(X, E_D)$, we define

$$\overline{\partial}_X s := [\tilde{X}, \tilde{s}] \mod D, \tag{A.3}$$

where $X \in \Gamma^\infty(X, T_{0,1}X)$, $\tilde{X}$ is the unique lift of X to a section of D, and $\tilde{s}$ is any lift of s to a section of $D^\perp$. The Jacobi identity for the Courant bracket implies that it induces a

Courant bracket on the holomorphic sections of E_D. In this way, we are able to describe the map

$$H^3(M,\mathbb{C}) \ni [E] \overset{D}{\longmapsto} [E_D] \in H^1(\Omega^{2,\mathrm{cl}}(X)),$$

without choosing splittings.

Just as in the case of symplectic reduction, where a Lagrangian submanifold may pass to the symplectic quotient, we may also reduce Dirac structures from E to E_D. The reduction of Dirac structures proceeds as follows [8].

Proposition A.6. *Let $L \subset E$ be a Dirac structure such that $L \cap D^\perp$ has constant rank and L is D-invariant, in the sense $[\mathcal{O}(D), \mathcal{O}(L)] \subset \mathcal{O}(L)$. Then the subbundle $L_D \subset E_D$, defined by*

$$L_D := \frac{L \cap D^\perp + D}{D}, \tag{A.4}$$

is holomorphic with respect to the induced holomorphic structure (A.3) on E_D and defines a holomorphic Dirac structure in E_D called the reduction of L.

B Gerbe Connections in Dirac Geometry

In this section we will review the relation between $\mathbb{C}^*$-gerbes and Courant algebroids described by Ševera [46], and extend it in two ways: we use gerbe connections over general Lie algebroids, and more meaningfully, we explain the relation between Dirac structures and gerbe connections. We take the Čech approach of [7,11,27] to the description of gerbes, omitting discussions of refinements of covers, for convenience.

Let M be a smooth real or complex manifold, where $\mathcal{O}_M$ denotes the sheaf of complex-valued functions (smooth or holomorphic, respectively). Choose an open covering $\{U_i\}$, and let G be a $\mathbb{C}^*$-gerbe which is locally trivialized over this covering, so that it is given by the data $\{L_{ij}, \theta_{ijk}\}$, where L_{ij} are (smooth or holomorphic) complex line bundles over U_{ij}, chosen so that L_{ij} is dual to L_{ji}, and $\theta_{ijk} : L_{ij} \otimes L_{jk} \longrightarrow L_{ik}$ are isomorphisms of line bundles over U_{ijk} such that on quadruple overlaps U_{ijkl} we have the coherence condition:

$$\theta_{ikl} \circ (\theta_{ijk} \otimes \mathrm{id}) = \theta_{ijl} \circ (\mathrm{id} \otimes \theta_{jkl}).$$

We now introduce the notion of a gerbe connection over an arbitrary Lie algebroid A. Let $(A, a, [\cdot,\cdot])$ be a complex Lie algebroid on M, where $[\cdot,\cdot]$ is the Lie bracket on the sheaf of sections of A, and $a : A \longrightarrow \mathrm{Der}(\mathcal{O}_M)$ is the bracket-preserving bundle map to the tangent bundle, usually called the anchor. We will use the notation $(\Omega^\bullet_A, d_A)$ to denote the associated de Rham complex of A. Note that if A is a holomorphic Lie algebroid, the anchor maps to the holomorphic tangent bundle, whereas in the smooth case it maps to $TM \otimes \mathbb{C}$.

Definition B.1. *An A-connection on a line bundle L is a differential operator*

$$\partial : \mathcal{O}(L) \longrightarrow \mathcal{O}(A^* \otimes L),$$

such that $\partial(fs) = (d_A f) \otimes s + f\partial s$, for $f \in \mathcal{O}_M$ and $s \in \mathcal{O}(L)$. As with usual connections, ∂ has a curvature tensor $\partial^2 = F_\partial \in \Omega^2_A(M)$ such that $d_A F_\partial = 0$. When $F_\partial = 0$, we say that L is flat over A, or that L is an A-module.

Definition B.2. *An A-connection (∂, B) on the gerbe G defined by $\{L_{ij}, \theta_{ijk}\}$ is given as follows. The first component, ∂, called the 0-connection, is a family of A-connections ∂_{ij} on L_{ij} with $\partial_{ji} = \partial_{ij}^*$ and such that θ_{ijk} is flat in the induced connection:*

$$\theta_{ijk} \circ (\partial_{ij} \otimes 1 + 1 \otimes \partial_{jk}) = \partial_{ik} \circ \theta_{ijk}. \tag{B.1}$$

The second component, B, called the 1-connection, is a collection $\{B_i \in \Omega_A^2(U_i)\}$ satisfying $B_j - B_i = F_{\partial_{ij}}$. The global 3-form $H \in \Omega_A^3(M)$ defined by $H|_{U_i} = d_A B_i$ is called the curving of the connection, and satisfies $d_A H = 0$. When $H = 0$, we say that G is flat over A.

An equivalence of gerbes $S : G \to G'$ may be described as an object S in $G^* \otimes G'$, which in a local trivialization is given by $\{L_i, m_{ij}\}$, where L_i are line bundles over U_i and m_{ij} are isomorphisms

$$m_{ij} : L_i \longrightarrow (L_{ij}^* \otimes L_{ij}') \otimes L_j.$$

If G, G' are equipped with A-connections $(\partial, B), (\partial', B')$, then to promote S to an equivalence of gerbes with connection is to equip L_i with A-connections ∂_i such that

$$m_{ij*}(\partial_i) = \partial_{ij}^* + \partial_{ij}' + \partial_j \quad \text{and} \quad B_i - B_i' = F_{\partial_i}.$$

An auto-equivalence with connection is then simply a line bundle with connection (L, ∂); its action is only seen by the data defining the 1-connection, via $B_i \mapsto B_i - F_\partial|_{U_i}$. Gerbes with A-connections are classified up to equivalence by the hypercohomology group

$$\mathbb{H}^2(\, \mathcal{O}_M^* \xrightarrow{d_A \log} \Omega_A^1 \xrightarrow{d_A} \Omega_A^2 \,).$$

As in the case of holomorphic vector bundles, where the existence of a holomorphic connection is obstructed by the Atiyah class, the existence of an A-connection on a gerbe is obstructed in general. We now briefly summarize the treatment of the obstructions given in [11].

Arbitrarily choose A-connections ∂_{ij} on L_{ij}, so that $\theta_{ijk} \in \mathcal{O}(L_{ik} \otimes L_{kj} \otimes L_{ji})$ is not necessarily flat for the induced connection ∂_{ikj}:

$$\partial_{ikj}\theta_{ijk} = A_{ijk} \otimes \theta_{ijk}.$$

This defines a Čech cocycle $\{A_{ijk} \in \Omega_A^1(U_{ijk})\}$, which represents the 0-Atiyah class $\alpha_0 = [A_{ijk}] \in H^2(\Omega_A^1)$ obstructing the existence of a 0-connection on the gerbe. If $\alpha_0 = 0$, then there exists a 0-connection $\{\partial_{ij}\}$, and the curvatures $\{F_{\partial_{ij}} \in \Omega_A^2(U_{ij})\}$ define the 1-Atiyah class $\alpha_1 \in H^1(\Omega_A^2)$ obstructing the existence of a 1-connection.

The following is an example of how A-connections on gerbes may be used. It is an analog of the well-known result for complex vector bundles that a flat partial (0, 1)-connection induces a holomorphic structure on the bundle. It is essentially a realization of the following isomorphism:

$$H^2(\mathcal{O}_{\text{hol}}^*) \cong \mathbb{H}^2(\, \mathcal{O}_\infty^* \xrightarrow{\bar\partial \log} \Omega^{0,1} \xrightarrow{\bar\partial} \Omega^{0,2} \xrightarrow{\bar\partial} \Omega^{0,3} \,).$$

Theorem B.3. *Let G be a smooth $\mathbb{C}^*$-gerbe over a complex manifold M, and let $A = T_{0,1}M$ be the Dolbeault Lie algebroid. The choice of a flat A-connection on G naturally endows the gerbe with a holomorphic structure.*

Our purpose in introducing connections on gerbes is twofold. First, taking A to be the tangent bundle, we associate, following [28,46], a canonical Courant algebroid E_∂ to a 0-connection ∂ on a gerbe. Second, we show that any Dirac structure $D \subset E_\partial$ induces a flat D-connection on the gerbe.

The Courant algebroid of a Gerbe with 0-connection

Theorem B.4. *To every 0-connection ∂ (for the complexified or holomorphic tangent bundle) on a gerbe G, there is a canonically associated Courant algebroid E_∂, with isomorphism class given by the map*

$$\mathbb{H}^2(\ \mathcal{O}_M^* \xrightarrow{d\log} \Omega^1\ ) \xrightarrow{d} \mathbb{H}^2(\ 0 \longrightarrow \Omega^2 \xrightarrow{d} \Omega^3 \xrightarrow{d} \Omega^4\ ) = H^1(\Omega^{2,\mathrm{cl}}). \tag{B.2}$$

Furthermore, global splittings of $E_\partial \xrightarrow{\pi} TM$ with isotropic image correspond bijectively with 1-connections on (G, ∂). A connection is flat if and only if the corresponding splitting is involutive.

Proof. Let the 0-connection be given by $a = \{\partial_{ij}\}$ in a local trivialization of the gerbe, and define the Courant algebroid E_a by gluing $\mathbb{T}U_i$ to $\mathbb{T}U_j$ using the B-field gauge transformation by $F_{\partial_{ij}}$, given by

$$e^{F_{\partial_{ij}}} = \begin{pmatrix} 1 & 0 \\ F_{\partial_{ij}} & 1 \end{pmatrix},$$

which satisfies the cocycle condition due to Eq. B.1. We must check that the Courant algebroid is independent of a. Change the local trivialization, using local line bundles with connection $g = \{L_i, D_i\}$, so that the 0-connection is given by the collection $a^g = \{\partial_{ij} + D_i + D_j^*\}$ of connections on $L_{ij} \otimes L_i \otimes L_j^*$. Hence E_{a^g} is constructed using $F_{\partial_{ij}} + F_{D_i} - F_{D_j}$. But then we obtain a map $\psi_g : E_a \longrightarrow E_{a^g}$ defined by $\psi_g|_{U_i} = e^{F_{D_i}}$, which is an isomorphism of Courant algebroids because it intertwines the gluing maps with isomorphisms of the Courant structure, i.e. the following diagram commutes:

$$\begin{array}{ccc} \mathbb{T}U_i & \xrightarrow{e^{F_{\partial_{ij}}}} & \mathbb{T}U_j \\ {\scriptstyle e^{F_{D_i}}}\downarrow & & \downarrow{\scriptstyle e^{F_{D_j}}} \\ \mathbb{T}U_i & \xrightarrow[e^{F_{\partial_{ij}}+F_{D_i}-F_{D_j}}]{} & \mathbb{T}U_j \end{array}$$

Functoriality follows from the fact that if $g = g_1 g_2$ is the tensor product of local line bundles with connection then $\psi_g = \psi_{g_1}\psi_{g_2}$. Hence we obtain a well-defined Courant algebroid E_∂ associated to the 0-connection.

Trivializing the line bundles L_{ij} so that the gerbe is given by data $g_{ijk} \in \mathcal{O}^*(U_{ijk})$ and ∂_{ij} is given by connection 1-forms A_{ij}, we see that the gluing 2-forms for the Courant algebroid are simply dA_{ij}; hence $[E_\partial]$ is indeed given by $d[(G, \partial)]$ as in (B.2).

A 1-connection B on (G, ∂) consists of 2-forms $B_i \in \Omega^2(U_i)$ with $B_j - B_i = F_{\partial_{ij}}$, data which determines a splitting s_B of $E_a \xrightarrow{\pi} TM$ defined by $s_B|_{TU_i}(X) = X + i_X B_i \in \mathbb{T}U_i$: clearly $e^{F_{\partial_{ij}}} \circ s_B|_{TU_i} = s_B|_{TU_j}$ on U_{ij}, rendering s_B well-defined. The map

$B \mapsto s_B$ is clearly a bijection, since isotropic splittings of $\mathbb{T}U_i \longrightarrow TU_i$ are simply graphs of 2-forms. The bijection is functorial because a change of local trivialization $g = \{L_i, D_i\}$ maps $B = \{B_i\}$ to $B^g = \{B_i - F_{D_i}\}$, so that the following diagram commutes:

$$\begin{array}{ccc} E_a & \xleftarrow{s_B} & TM \\ \downarrow{\scriptstyle \psi_g} & & \downarrow{\scriptstyle \mathrm{id}} \\ E_{a^g} & \xleftarrow{s_{B^g}} & TM \end{array}$$

Finally, the graph of a 2-form in $\mathbb{T}M$ is involutive if and only if it is closed, hence the splitting s_B is involutive if and only if the gerbe is flat over TM. □

Theorem B.4 is stated for the smooth complexified tangent bundle or the holomorphic tangent bundle. To obtain smooth real Courant algebroids, we equip the gerbe with a Hermitian structure [27] and require the 0-connection ∂ to be unitary, as follows.

Definition B.5. *A Hermitian structure on the gerbe G defined by $\{L_{ij}, \theta_{ijk}\}$ is given by a family of Hermitian metrics $h = \{h_{ij}\}$ on the complex line bundles L_{ij}, such that θ_{ijk} is unitary. A connection on G defined by $\{\nabla_{ij}, B_i\}$ is unitary when ∇_{ij} are unitary connections (with* real *curvatures, by convention) and B_i are real.*

The analog of Theorem B.4 for Hermitian gerbes is then

Corollary B.6. *To every unitary 0-connection $\nabla = \{\nabla_{ij}\}$ (over the real tangent bundle) on a Hermitian gerbe (G, h), there is a canonically associated real Courant algebroid E_∇, with isomorphism class given by the map*

$$\mathbb{H}^2(\ \mathcal{O}(U(1)) \xrightarrow{-id\log} \Omega^1_{\mathbb{R}}\ ) \xrightarrow{d} H^1(\Omega^{2,\mathrm{cl}}_{\mathbb{R}}) = H^3(M, \mathbb{R}). \tag{B.3}$$

The correspondence between splittings and 1-connections is as before.

Dirac structures and gerbes. The presence of a Dirac structure in the Courant algebroid associated to a gerbe with 0-connection imposes a strong constraint on the gerbe, which we now make precise.

Theorem B.7. *Let E_∂ be the Courant algebroid associated to a gerbe with 0-connection (G, ∂). Given an involutive isotropic subbundle $D \subset E_\partial$, the gerbe G inherits a canonical flat D-connection.*

Proof. The restriction of $\pi : E_\partial \longrightarrow TM$ as well as the Courant bracket to D gives it the structure of a Lie algebroid, and by choosing a local trivialization for (G, ∂), we immediately obtain a 0-connection ∂^D over D by composition:

$$\partial^D_{ij} := \pi|^*_D \circ \partial_{ij}.$$

To obtain the 1-connection B over D, write the inclusion $D \subset E_\partial$ locally, as involutive isotropic subbundles $D_i \subset \mathbb{T}U_i$ such that $e^{F_{\partial_{ij}}} D_i = D_j$. Then consider the antisymmetric pairing on $\mathbb{T}U_i$:

$$\langle X + \xi, Y + \eta \rangle_- := \frac{1}{2}(\xi(Y) - \eta(X)).$$

This restricts to D_i and determines 2-forms $B_i \in \Omega^2_D(U_i)$. The gluing condition $e^{F_{\partial_{ij}}} D_i = D_j$ implies that $B_i - B_j = \pi|_D^* F_{ij} = F_{\partial^D_{ij}}$, so that $\{\partial^D_{ij}, B_i\}$ is indeed a D-connection.

We now check that the D-connection is independent of the local trivialization used to define it. In a local trivialization differing from the initial one by the local line bundles with connection $g = \{L_i, \partial_i\}$, the 0-connection over D is given by $\pi|_D^*(\partial_{ij} + \partial_i + \partial_j^*)$, and the effect on E_∂ is via the isomorphism ψ_g, which sends D_i to $e^{F_{\partial_i}} D_i$, so that the restriction of the antisymmetric pairing to D_i yields $B_i + \pi|_D^* F_{\partial_i}$. The resulting expression for the D-connection is precisely that obtained by changing the local trivialization of (G, ∂^D) by the local line bundles with D-connection $g^D := \{L_i, \pi|_D^* \partial_i\}$. The naturality of the map $g \mapsto g^D$ ensures that (G, ∂^D, B) is canonically defined.

Finally, the curving of the D-connection may be computed using a general property of the Courant bracket implicit in Theorem 2.3.6. of [13], namely that the restriction of $\langle\cdot,\cdot\rangle_-$ to any isotropic integrable subbundle $D_i \subset \mathbb{T}M$ is closed with respect to the algebroid differential. □

We now show that the above theorem may be used to endow a gerbe with a holomorphic structure, in such a way that the resulting holomorphic gerbe inherits a holomorphic 0-connection. At the level of Courant algebroids, this is a reduction procedure as in [8], whereby a smooth complex Courant algebroid "reduces" to a holomorphic one.

Theorem B.8. *Let (G, ∇) be a smooth $\mathbb{C}^*$-gerbe with 0-connection over a complex manifold X, and let $D \subset E_\nabla$ be a lifting of $T_{0,1}X$ to the complex Courant algebroid associated to ∇, in the sense of Definition A.1. Then G inherits a holomorphic structure. Furthermore, G inherits a canonical holomorphic 0-connection ∂.*

Proof. By Theorem B.7, the presence of $D \subset E_\nabla$ immediately endows G with a flat D-connection. Since D is isomorphic to $T_{0,1}X$, the gerbe G is endowed with a holomorphic structure by Theorem B.3. What remains is to show G inherits a holomorphic 0-connection.

Choose a local trivialization in which the gerbe with 0-connection is given by $\{L_{ij}, \theta_{ijk}, \nabla_{ij}\}$, the Courant algebroid E_∇ is given as in Theorem B.4, and $D_i = D|_{U_i}$ is given by the graph of $\theta_i \in \Omega^{(1,1)+(0,2)}(U_i)$, so that involutivity is the condition

$$(d\theta_i)^{(1,2)+(0,3)} = 0.$$

Since $F_{\nabla_{ij}}$ must glue D_i to D_j, we have

$$(F_{\nabla_{ij}})^{(1,1)+(0,2)} = \theta_j - \theta_i.$$

Refining the cover if necessary, choose $\alpha = \{\alpha_i \in \Omega^{(1,0)+(0,1)}(U_i)\}$ such that

$$(d\alpha_i)^{(1,1)+(0,2)} = \theta_i.$$

Changing the local trivialization by the local line bundles with connection $(U_i \times \mathbb{C}, d + \alpha_i)$, the 0-connection has the expression $\nabla_{ij} + \alpha_i - \alpha_j$, which has curvature of type $(2,0)$. This defines a holomorphic gerbe with holomorphic 0-connection $(G_\alpha, \partial_\alpha)$, which a priori depends on α. But two choices α, α' of potential for $\{\theta_i\}$, as above, give rise naturally to the local holomorphic line bundles $L_i := (U_i \times \mathbb{C}, \overline{\partial} + \alpha'^{0,1}_i - \alpha^{0,1}_i)$, with holomorphic connections given by $\partial_i := \partial + \alpha'^{1,0}_i - \alpha^{1,0}_i$. The local holomorphic line bundles with holomorphic connections (L_i, ∂_i) then define an equivalence

$$(L_i, \partial_i) : (\mathcal{G}_\alpha, \partial_\alpha) \longrightarrow (\mathcal{G}_{\alpha'}, \partial_{\alpha'}).$$

We omit the straightforward verification that the resulting holomorphic gerbe with 0-connection is independent of the choices made. □

Remark B.9. A 1-connection on a gerbe with 0-connection ∇ is a maximal isotropic splitting of the Courant algebroid E_∇; for this reason we may view the lifting $D \subset E_\nabla$ of the Theorem as a partial 1-connection on the gerbe.

Example B.10. Consider the Hopf surface X from Example 2.3, viewed as an elliptic fibration over $\mathbb{C}P^1$ via the map $(x_1, x_2) \mapsto [x_1 : x_2]$. Choose affine charts (U_0, z_0), (U_1, z_1) for the base $\mathbb{C}P^1$, and write X as the gluing of $(z_0, w_0) \in U_0 \times (\mathbb{C}^*/\mathbb{Z})$ to $(z_1, w_1) \in U_1 \times (\mathbb{C}^*/\mathbb{Z})$ by the map

$$(z_0, w_0) \longmapsto (1/z_0, z_0 w_0).$$

On $U_0 \cap U_1$ we have the following real 2-form

$$F_{01} = \frac{-1}{4\pi}\left(\frac{dz_0 \wedge dw_0}{z_0 w_0} + \frac{d\bar{z}_0 \wedge d\bar{w}_0}{\bar{z}_0 \bar{w}_0}\right).$$

The only nonvanishing period of this 2-form is for the cycle $S^1 \times S^1 \subset \mathbb{C}^* \times \mathbb{C}^*$, which yields $\frac{-1}{4\pi} 2(2\pi i)(2\pi i) = 2\pi$. Since F_{01} is integral, we may "prequantize" it, viewing it as the curvature of a Hermitian line bundle $(L_{01}, h_{01}, \nabla_{01})$ with unitary connection ∇_{01}. This defines the structure of a Hermitian gerbe with 0-connection ∇ over X, such that the associated Courant algebroid E_∇ is precisely that from Example 2.4.

To describe a lifting of $T_{0,1}X$ to the Courant algebroid E_∇, it is convenient to choose a 1-connection

$$B_i = \frac{1}{4\pi}(\partial K_i \wedge \partial \log w_i + \overline{\partial} K_i \wedge \overline{\partial} \log \bar{w}_i),$$

where $K_i = \log(1 + z_i \bar{z}_i)$ are the usual Kähler potentials for the Fubini-Study metric on $\mathbb{C}P^1$. Computing the global real 3-form $H = dB_i$, we obtain the $(1, 2) + (2, 1)$-form

$$H = \frac{-1}{8\pi} dd^c K_i \wedge d^c \log(w_i \bar{w}_i).$$

Observe that $H = d^c\omega$, for the $(1, 1)$-form

$$\omega = \frac{-i}{4\pi}(\overline{\partial} K_0 \wedge \partial \log w - \partial K_0 \wedge \overline{\partial} \log \bar{w}), \tag{B.4}$$

with the significance that $H^{1,2} = \overline{\partial}(i\omega)$, which is precisely the condition (A.2) that $i\omega$ defines a lifting of $T_{0,1}X$. As a consequence of choosing ω, we obtain a canonical holomorphic structure on the gerbe, as follows. Returning to the Čech description, the lifting defined by ω is described by the local forms

$$\theta_i = B_i^{0,2} - i\omega|_{U_i}.$$

Our open cover is such that $\theta_i^{0,2}$ is $\overline{\partial}$-exact, namely

$$\theta_i^{0,2} = \overline{\partial}(\frac{1}{4\pi} K_i \wedge \overline{\partial} \log \bar{w}_i).$$

Following Theorem B.8, we perform a gauge transformation by $a = \{a_i = \frac{1}{4\pi} K_i d \log(w_i \bar{w}_i)\}$; the new unitary connection $\nabla_{01}^a = \nabla_{01} + a_0 - a_1$ has curvature of type (1, 1) given by

$$F_{01}^a = \frac{1}{4\pi} \left(\frac{dz_0 \wedge d\bar{w}_0}{z_0 \bar{w}_0} + \frac{d\bar{z}_0 \wedge dw_0}{\bar{z}_0 w_0} \right),$$

so that $\nabla_{0,1}^a$ is indeed a holomorphic structure on the gerbe. After the gauge transformation, the lifting is described by

$$\theta_i^a = \theta_i - (da_i)^{(1,1)+(0,2)} = \frac{-1}{2\pi} \overline{\partial} K_i \wedge \partial \log w_i.$$

While θ_i^a is $\overline{\partial}$-closed, it is not exact; therefore, to explicitly describe the holomorphic 0-connection on the gerbe we would need to refine the cover. Nevertheless, the associated holomorphic Courant algebroid may be easily constructed; by the prescription in Theorem A.5, it is given by the following holomorphic (2, 0)-form:

$$\begin{aligned} \mathcal{B}_{01} &= (B_1^{2,0} + \partial a_1^{1,0} - B_0^{2,0} - \partial a_0^{1,0}) \\ &= \frac{-1}{2\pi} (z_0 w_0)^{-1} dz_0 \wedge dw_0. \end{aligned}$$

In this way, we recover the holomorphic Courant algebroid studied in Example 2.3.

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