3.6 Partitions of unity and Whitney embedding

Partitions of unity allow us to *go from local to global*, i.e. to build a global object on a manifold by building it on each open set of a cover, smoothly tapering each local piece so it is compactly supported in each open set, and then taking a sum over open sets. This is a very flexible operation which uses the properties of smooth functions—it will not work for complex manifolds, for example. Our main example of such a passage from local to global is to build a global map from a manifold to $\mathbb{R}^N$ which is an embedding, a result first proved by Whitney.

Definition 3.44. A collection of subsets $\{U_\alpha\}$ of the topological space M is called *locally finite* when each point $x \in M$ has a neighbourhood V intersecting only finitely many of the U_α.

Definition 3.45. A covering $\{V_\alpha\}$ is a *refinement* of the covering $\{U_\beta\}$ when each V_α is contained in some U_β.

Lemma 3.46. *Any open covering $\{A_\alpha\}$ of a topological manifold has a countable, locally finite refinement $\{(U_i, \varphi_i)\}$ by coordinate charts such that $\varphi_i(U_i) = B(0,3)$ and $\{V_i = \varphi_i^{-1}(B(0,1))\}$ is still a covering of M. We will call such a cover a* regular *covering. In particular, any topological manifold is* paracompact *(i.e. every open cover has a locally finite refinement)*

Proof. If M is compact, the proof is easy: choosing coordinates around any point $x \in M$, we can translate and rescale to find a covering of M by a refinement of the type desired, and choose a finite subcover, which is obviously locally finite.

For a general manifold, we note that by second countability of M, there is a countable basis of coordinate neighbourhoods and each of these charts is a countable union of open sets P_i with $\overline{P_i}$ compact. Hence M has a countable basis $\{P_i\}$ such that $\overline{P_i}$ is compact.

Using these, we may define an increasing sequence of compact sets which exhausts M: let $K_1 = \overline{P}_1$, and

$$K_{i+1} = \overline{P_1 \cup \cdots \cup P_r},$$

where $r > 1$ is the first integer with $K_i \subset P_1 \cup \cdots \cup P_r$.

Now note that M is the union of ring-shaped sets $K_i \backslash K_{i-1}^\circ$, each of which is compact. If $p \in A_\alpha$, then $p \in K_{i+1} \backslash K_i^\circ$ for some i. Now choose a coordinate neighbourhood $(U_{p,\alpha}, \varphi_{p,\alpha})$ with $U_{p,\alpha} \subset K_{i+2} \backslash K_{i-1}^\circ$ and $\varphi_{p,\alpha}(U_{p,\alpha}) = B(0,3)$ and define $V_{p,\alpha} = \varphi^{-1}(B(0,1))$.

Letting p, α vary, these neighbourhoods cover the compact set $K_{i+1} \backslash K_i^\circ$ without leaving the band $K_{i+2} \backslash K_{i-1}^\circ$. Choose a finite subcover $V_{i,k}$ for each i. Then $(U_{i,k}, \varphi_{i,k})$ is the desired locally finite refinement. □

Definition 3.47. A smooth partition of unity is a collection of smooth non-negative functions $\{f_\alpha : M \longrightarrow \mathbb{R}\}$ such that

i) $\{\mathrm{supp} f_\alpha = \overline{f_\alpha^{-1}(\mathbb{R} \backslash \{0\})}\}$ is locally finite,

ii) $\sum_\alpha f_\alpha(x) = 1 \ \ \forall x \in M$, hence the name.

A partition of unity is *subordinate* to an open cover $\{U_i\}$ when $\forall \alpha$, $\mathrm{supp} f_\alpha \subset U_i$ for some i.

Theorem 3.48. *Given a regular covering $\{(U_i, \varphi_i)\}$ of a manifold, there exists a partition of unity $\{f_i\}$ subordinate to it with $f_i > 0$ on V_i and supp$f_i \subset \varphi_i^{-1}(\overline{B(0,2)})$.*

Proof. A *bump function* is a smooth non-negative real-valued function $\tilde{g}$ on $\mathbb{R}^n$ with $\tilde{g}(x) = 1$ for $||x|| \leq 1$ and $\tilde{g}(x) = 0$ for $||x|| \geq 2$. For instance, take

$$\tilde{g}(x) = \frac{h(2 - ||x||)}{h(2 - ||x||) + h(||x|| + 1)},$$

for $h(t)$ given by $e^{-1/t}$ for $t > 0$ and 0 for $t < 0$.

Having this bump function, we can produce non-negative bump functions on the manifold $g_i = \tilde{g} \circ \varphi_i$ which have support supp$g_i \subset \varphi_i^{-1}(\overline{B(0,2)})$ and take the value $+1$ on $\overline{V_i}$. Finally we define our partition of unity via

$$f_i = \frac{g_i}{\sum_j g_j}, \quad i = 1, 2, \ldots.$$

□

We now investigate the embedding of arbitrary smooth manifolds as regular submanifolds of $\mathbb{R}^k$. We shall first show by a straightforward argument that any smooth manifold may be embedded in some $\mathbb{R}^N$ for some sufficiently large N. We will then explain how to cut down on N and approach the optimal $N = 2 \dim M$ which Whitney showed (we shall reach $2 \dim M + 1$ and possibly at the end of the course, show $N = 2 \dim M$.)

Theorem 3.49 (Compact Whitney embedding in $\mathbb{R}^N$)**.** *Any* compact *manifold may be embedded in $\mathbb{R}^N$ for sufficiently large N.*

Proof. Let $\{(U_i \supset V_i, \varphi_i)\}_{i=1}^k$ be a *finite* regular covering, which exists by compactness. Choose a partition of unity $\{f_1, \ldots, f_k\}$ as in Theorem 3.48 and define the following "zoom-in" maps $M \longrightarrow \mathbb{R}^{\dim M}$:

$$\tilde{\varphi}_i(x) = \begin{cases} f_i(x)\varphi_i(x) & x \in U_i, \\ 0 & x \notin U_i. \end{cases}$$

Then define a map $\Phi : M \longrightarrow \mathbb{R}^{k(\dim M + 1)}$ which zooms simultaneously into all neighbourhoods, with extra information to guarantee injectivity:

$$\Phi(x) = (\tilde{\varphi}_1(x), \ldots, \tilde{\varphi}_k(x), f_1(x), \ldots, f_k(x)).$$

Note that $\Phi(x) = \Phi(x')$ implies that for some i, $f_i(x) = f_i(x') \neq 0$ and hence $x, x' \in U_i$. This then implies that $\varphi_i(x) = \varphi_i(x')$, implying $x = x'$. Hence Φ is injective.

We now check that $D\Phi$ is injective, which will show that it is an injective immersion. At any point x the differential sends $v \in T_xM$ to the following vector in $\mathbb{R}^{\dim M} \times \cdots \times \mathbb{R}^{\dim M} \times \mathbb{R} \times \cdots \times \mathbb{R}$.

$$(Df_1(v)\varphi_1(x)+f_1(x)D\varphi_1(v), \ldots, Df_k(v)\varphi_k(x)+f_k(x)D\varphi_1(v), Df_1(v), \ldots, Df_k(v)$$

But this vector cannot be zero. Hence we see that Φ is an immersion.

But an injective immersion from a compact space must be an embedding: view Φ as a bijection onto its image. We must show that Φ^{-1} is continuous, i.e. that Φ takes closed sets to closed sets. If $K \subset M$ is closed, it is also compact and hence $\Phi(K)$ must be compact, hence closed (since the target is Hausdorff). □

Theorem 3.50 (Compact Whitney embedding in $\mathbb{R}^{2n+1}$). *Any compact n-manifold may be embedded in $\mathbb{R}^{2n+1}$.*

Proof. Begin with an embedding $\Phi : M \longrightarrow \mathbb{R}^N$ and assume $N > 2n+1$. We then show that by projecting onto a hyperplane it is possible to obtain an embedding to $\mathbb{R}^{N-1}$.

A vector $v \in S^{N-1} \subset \mathbb{R}^N$ defines a hyperplane (the orthogonal complement) and let $P_v : \mathbb{R}^N \longrightarrow \mathbb{R}^{N-1}$ be the orthogonal projection to this hyperplane. We show that the set of v for which $\Phi_v = P_v \circ \Phi$ fails to be an embedding is a set of measure zero, hence that it is possible to choose v for which Φ_v is an embedding.

Φ_v fails to be an embedding exactly when Φ_v is not injective or $D\Phi_v$ is not injective at some point. Let us consider the two failures separately:

If v is in the image of the map $\beta_1 : (M \times M)\backslash\Delta_M \longrightarrow S^{N-1}$ given by

$$\beta_1(p_1, p_2) = \frac{\Phi(p_2) - \Phi(p_1)}{||\Phi(p_2) - \Phi(p_1)||},$$

then Φ_v will fail to be injective. Note however that β_1 maps a $2n$-dimensional manifold to a $N-1$-manifold, and if $N > 2n+1$ then baby Sard's theorem implies the image has measure zero.

The immersion condition is a local one, which we may analyze in a chart (U, φ). Φ_v will fail to be an immersion in U precisely when v coincides with a vector in the normalized image of $D(\Phi \circ \varphi^{-1})$ where

$$\Phi \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^n \longrightarrow \mathbb{R}^N.$$

Hence we have a map (letting $N(w) = ||w||$)

$$\frac{D(\Phi \circ \varphi^{-1})}{N \circ D(\Phi \circ \varphi^{-1})} : U \times S^{n-1} \longrightarrow S^{N-1}.$$

The image has measure zero as long as $2n-1 < N-1$, which is certainly true since $2n < N-1$. Taking union over countably many charts, we see that immersion fails on a set of measure zero in S^{N-1}.

Hence we see that Φ_v fails to be an embedding for a set of $v \in S^{N-1}$ of measure zero. Hence we may reduce N all the way to $N = 2n+1$. □

Corollary 3.51. *We see from the proof that if we do not require injectivity but only that the manifold be* immersed *in* $\mathbb{R}^N$*, then we can take* $N = 2n$ *instead of* $2n+1$.

Theorem 3.52 (noncompact Whitney embedding in $\mathbb{R}^{2n+1}$). *Any smooth* n*-manifold may be embedded in* $\mathbb{R}^{2n+1}$ *(or immersed in* $\mathbb{R}^{2n}$*).*

Proof. We saw that any manifold may be written as a countable union of increasing compact sets $M = \cup K_i$, and that a regular covering $\{(U_{i,k} \supset V_{i,k}, \varphi_{i,k})\}$ of M can be chosen so that for fixed i, $\{V_{i,k}\}_k$ is a finite cover of $K_{i+1} \backslash K_i^\circ$ and each $U_{i,k}$ is contained in $K_{i+2} \backslash K_{i-1}^\circ$.

This means that we can express M as the union of 3 open sets W_0, W_1, W_2, where

$$W_j = \bigcup_{i \equiv j (\mathrm{mod} 3)} (\cup_k U_{i,k}).$$

Each of the sets $R_i = \cup_k U_{i,k}$ may be injectively immersed in $\mathbb{R}^{2n+1}$ by the argument for compact manifolds, since they have a finite regular cover. Call these injective immersions $\Phi_i : R_i \longrightarrow \mathbb{R}^{2n+1}$. The image $\Phi_i(R_i)$ is bounded since all the charts are, by some radius r_i. The open sets R_i, $i \equiv j (\mathrm{mod} 3)$ for fixed j are disjoint, and by translating each Φ_i, $i \equiv j(\mathrm{mod}3)$ by an appropriate constant, we can ensure that their images in $\mathbb{R}^{2n+1}$ are disjoint as well.

Let $\Phi_i' = \Phi_i + (2(r_{i-1} + r_{i-2} + \cdots) + r_i)\overrightarrow{e}_1$. Then $\Psi_j = \cup_{i \equiv j(\mathrm{mod}3)} \Phi_i' : W_j \longrightarrow \mathbb{R}^{2n+1}$ is an embedding.

Now that we have injective immersions Ψ_0, Ψ_1, Ψ_2 of W_0, W_1, W_2 in $\mathbb{R}^{2n+1}$, we may use the original argument for compact manifolds: Take the partition of unity subordinate to $U_{i,k}$ and resum it, obtaining a 3-element partition of unity $\{f_1, f_2, f_3\}$, with $f_j = \sum_{i \equiv j(\mathrm{mod}3)} \sum_k f_{i,k}$. Then the map

$$\Psi = (f_1\Psi_1, f_2\Psi_2, f_3\Psi_3, f_1, f_2, f_3)$$

is an injective immersion of M into $\mathbb{R}^{6n+3}$. To see that it is in fact an embedding, note that any closed set $C \subset M$ may be written as a union of closed sets $C = C_1 \cup C_2 \cup C_3$, where $C_j = \cup_{i \equiv j(\mathrm{mod}3)}(C \cap K_{i+1} \backslash K_i^\circ)$ is a disjoint union of compact sets. Ψ is injective, hence C_j is mapped to a disjoint union of compact sets, hence a closed set. Then $\Psi(C)$ is a union of 3 closed sets, hence closed, as required.

Using projection to hyperplanes we may again reduce to $\mathbb{R}^{2n+1}$, but if we exclude all hyperplanes perpendicular to $\mathrm{Span}((e_1, 0, 0, 0, 0, 0), (0, e_1, 0, 0, 0, 0), (0, 0, e_1, 0, 0, 0))$, we obtain an injective immersion Ψ' which is *proper*, meaning that inverse images of compact sets are compact. This space of forbidden planes has measure zero as long as $N - 1 > 3$, so that we may reduce to $2n+1$ for $n > 1$. We leave as an exercise the $n = 1$ case (or see Bredon for a slightly different proof).

The fact that the resulting injective immersion Ψ' is proper implies that it is an embedding, by the closed map lemma, as follows. □

Lemma 3.53 (Closed map lemma for proper maps). *Let* $f : X \longrightarrow Y$ *be a proper continuous map of topological manifolds. Then* f *is a closed map.*

Proof. Let $K \subset X$ be closed; we show that $f(K)$ contains all its limit points and hence is closed. Let $y \in Y$ be a limit point for $f(K)$. Choose a precompact neighbourhood U of y, so that y is also a limit point of $f(K) \cap \overline{U}$. Since f is proper, $f^{-1}(\overline{U})$ is compact, and hence $K \cap f^{-1}(\overline{U})$ is compact as well. But then by continuity, $f(K \cap f^{-1}(\overline{U})) = f(K) \cap \overline{U}$ is compact, implying it is closed. Hence $y \in f(K) \cap \overline{U} \subset f(K)$, as required. $\square$

We now use Whitney embedding to prove the existence of tubular neighbourhoods for submanifolds of $\mathbb{R}^N$, a key point in proving genericity of transversality. Tubular neighbourhoods also exist for submanifolds of any manifold, but we leave this corollary for the reader.

If $Y \subset \mathbb{R}^N$ is an embedded submanifold, the normal space at $y \in Y$ is defined by $N_y Y = \{v \in \mathbb{R}^N \ : \ v \perp T_y Y\}$. The collection of all normal spaces of all points in Y is called the normal bundle:

$$NY = \{(y, v) \in Y \times \mathbb{R}^N \ : \ v \in N_y Y\}.$$

Proposition 3.54. *$NY \subset \mathbb{R}^N \times \mathbb{R}^N$ is an embedded submanifold of dimension N.*

Proof. Given $y \in Y$, choose coordinates $(u^1, \ldots u^N)$ in a neighbourhood $U \subset \mathbb{R}^N$ of y so that $Y \cap U = \{u^{n+1} = \cdots = u^N = 0\}$. Define $\Phi : U \times \mathbb{R}^N \longrightarrow \mathbb{R}^{N-n} \times \mathbb{R}^n$ via

$$\Phi(x, v) = (u^{n+1}(x), \ldots, u^N(x), \langle v, \tfrac{\partial}{\partial u^1}|_x\rangle, \ldots, \langle v, \tfrac{\partial}{\partial u^n}|_x\rangle),$$

so that $\Phi^{-1}(0)$ is precisely $NY \cap (U \times \mathbb{R}^N)$. We then show that 0 is a regular value: observe that, writing v in terms of its components $v^j \frac{\partial}{\partial x^j}$ in the standard basis for $\mathbb{R}^N$,

$$\langle v, \tfrac{\partial}{\partial u^i}|_x\rangle = \langle v^j \tfrac{\partial}{\partial x^j}, \tfrac{\partial x^k}{\partial u^i}(u(x)) \tfrac{\partial}{\partial x^k}|_x\rangle = \sum_{j=1}^N v^j \tfrac{\partial x^j}{\partial u^i}(u(x))$$

Therefore the Jacobian of Φ is the $((N-n)+n) \times (N+N)$ matrix

$$D\Phi(x) = \begin{pmatrix} \frac{\partial u^j}{\partial x^i}(x) & 0 \\ * & \frac{\partial x^j}{\partial u^i}(u(x)) \end{pmatrix}$$

The N rows of this matrix are linearly independent, proving Φ is a submersion. $\square$

The normal bundle NY contains $Y \cong Y \times \{0\}$ as a regular submanifold, and is equipped with a smooth map $\pi : NY \longrightarrow Y$ sending $(y, v) \mapsto y$. The map π is a surjective submersion and is the bundle projection. The vector spaces $\pi^{-1}(y)$ for $y \in Y$ are called the fibers of the bundle and NY is an example of a vector bundle.

We may take advantage of the embedding in $\mathbb{R}^N$ to define a smooth map $E : NY \longrightarrow \mathbb{R}^N$ via

$$E(x, v) = x + v.$$

Definition 3.55. A tubular neighbourhood of the embedded submanifold $Y \subset \mathbb{R}^N$ is a neighbourhood U of Y in $\mathbb{R}^N$ that is the diffeomorphic image under E of an open subset $V \subset NY$ of the form

$$V = \{(y, v) \in NY \ : \ |v| < \delta(y)\},$$

for some positive continuous function $\delta : M \longrightarrow \mathbb{R}$.

If $U \subset \mathbb{R}^N$ is such a tubular neighbourhood of Y, then there does exist a positive continuous function $\epsilon : Y \longrightarrow \mathbb{R}$ such that $U_\epsilon = \{x \in \mathbb{R}^N \ : \ \exists y \in Y \text{ with } |x - y| < \epsilon(y)\}$ is contained in U. This is simply

$$\epsilon(y) = \sup\{r \ : \ B(y, r) \subset U\},$$

which is continuous since $\forall \epsilon > 0, \exists x \in U$ for which $\epsilon(y) \leq |x - y| + \epsilon$. For any other $y' \in Y$, this is $\leq |y - y'| + |x - y'| + \epsilon$. Since $|x - y'| \leq \epsilon(y')$, we have $|\epsilon(y) - \epsilon(y')| \leq |y - y'| + \epsilon$.

Theorem 3.56 (Tubular neighbourhood theorem)**.** *Every regular submanifold of $\mathbb{R}^N$ has a tubular neighbourhood.*

Proof. First we show that E is a local diffeomorphism near $y \in Y \subset NY$. if ι is the embedding of Y in $\mathbb{R}^N$, and $\iota' : Y \longrightarrow NY$ is the embedding in the normal bundle, then $E \circ \iota' = \iota$, hence we have $DE \circ D\iota' = D\iota$, showing that the image of $DE(y)$ contains T_yY. Now if ι is the embedding of N_yY in $\mathbb{R}^N$, and $\iota' : N_yY \longrightarrow NY$ is the embedding in the normal bundle, then $E \circ \iota' = \iota$. Hence we see that the image of $DE(y)$ contains N_yY, and hence the image is all of $T_y\mathbb{R}^N$. Hence E is a diffeomorphism on some neighbourhood

$$V_\delta(y) = \{(y', v') \in NY \ : \ |y' - y| < \delta, \ |v'| < \delta\}, \quad \delta > 0.$$

Now for $y \in Y$ let $r(y) = \sup\{\delta \ : \ E|_{V_\delta(y)} \text{ is a diffeomorphism}\}$ if this is ≤ 1 and let $r(y) = 1$ otherwise. The function $r(y)$ is continuous, since if $|y - y'| < r(y)$, then $V_\delta(y') \subset V_{r(y)}(y)$ for $\delta = r(y) - |y - y'|$. This means that $r(y') \geq \delta$, i.e. $r(y) - r(y') \leq |y - y'|$. Switching y and y', this remains true, hence $|r(y) - r(y')| \leq |y - y'|$, yielding continuity.

Finally, let $V = \{(y, v) \in NY \ : \ |v| < \frac{1}{2}r(y)\}$. We show that E is injective on V. Suppose $(y, v), (y', v') \in V$ are such that $E(y, v) = E(y', v')$, and suppose wlog $r(y') \leq r(y)$. Then since $y + v = y' + v'$, we have

$$|y - y'| = |v - v'| \leq |v| + |v'| \leq \tfrac{1}{2}r(y) + \tfrac{1}{2}r(y') \leq r(y).$$

Hence y, y' are in $V_{r(y)}(y)$, on which E is a diffeomorphism. The required tubular neighbourhood is then $U = E(V)$. $\square$