2.6 Smooth maps between manifolds with boundary

We may also use the constant rank theorem to study manifolds with boundary.

Proposition 2.19. *Let M be a smooth n-manifold and $f : M \longrightarrow \mathbb{R}$ a smooth and proper real-valued function, and let a, b, with $a < b$, be regular values of f. Then $f^{-1}([a, b])$ is a cobordism between the closed $n-1$-manifolds $f^{-1}(a)$ and $f^{-1}(b)$.*

Proof. The pre-image $f^{-1}((a, b))$ is an open subset of M and hence a submanifold. Since p is regular for all $p \in f^{-1}(a)$, we may (by the constant rank theorem) find charts such that f is given near p by the linear map

$$(x_1, \ldots, x_m) \mapsto x_m. \tag{51}$$

Possibly replacing x_m by $-x_m$, we therefore obtain a chart near p for $f^{-1}([a, b])$ into H^m, as required. Proceed similarly for $p \in f^{-1}(b)$. □

Example 2.20. Using $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $(x_1, \ldots, x_n) \mapsto \sum x_i^2$, this gives a simple proof for the fact that the closed unit ball $\overline{B(0, 1)} = f^{-1}([-1, 1])$ is a manifold with boundary.

Example 2.21. Consider the C^∞ function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by $(x, y, z) \mapsto x^2 + y^2 - z^2$. Both $+1$ and -1 are regular values for this map, with pre-images given by 1- and 2-sheeted hyperboloids, respectively. Hence $f^{-1}([-1, 1])$ is a cobordism between hyperboloids of 1 and 2 sheets. In other words, it defines a cobordism between the disjoint union of two closed disks and the closed cylinder (each of which has boundary $S^1 \sqcup S^1$). Does this cobordism tell us something about the cobordism class of a connected sum?

Proposition 2.22. *Let $f : M \longrightarrow N$ be a smooth map from a manifold with boundary to the manifold N. Suppose that $q \in N$ is a regular value of f and also of $f|_{\partial M}$. Then the pre-image $f^{-1}(q)$ is a submanifold with boundary*[2]*. Furthermore, the boundary of $f^{-1}(q)$ is simply its intersection with ∂M.*

Proof. If $p \in f^{-1}(q)$ is not in ∂M, then as before $f^{-1}(q)$ is a submanifold in a neighbourhood of p. Therefore suppose $p \in \partial M \cap f^{-1}(q)$. Pick charts φ, ψ so that $\varphi(p) = 0$ and $\psi(q) = 0$, and $\psi f \varphi^{-1}$ is a map $U \subset H^m \longrightarrow \mathbb{R}^n$. Extend this to a smooth function $\tilde{f}$ defined in an open set $\tilde{U} \subset \mathbb{R}^m$ containing U. Shrinking $\tilde{U}$ if necessary, we may assume $\tilde{f}$ is regular on $\tilde{U}$. Hence $\tilde{f}^{-1}(0)$ is a submanifold of $\mathbb{R}^m$ of codimension n.

Now consider the real-valued function $\pi : \tilde{f}^{-1}(0) \longrightarrow \mathbb{R}$ given by the restriction of $(x_1, \ldots, x_m) \mapsto x_m$. $0 \in \mathbb{R}$ must be a regular value of π, since if not, then the tangent space to $\tilde{f}^{-1}(0)$ at 0 would lie completely in $x_m = 0$, which contradicts the fact that q is a regular point for $f|_{\partial M}$.

Hence, by Proposition 2.19, we have expressed $f^{-1}(q)$, in a neighbourhood of p, as a regular submanifold with boundary given by $\{\varphi^{-1}(x) \ : \ x \in \tilde{f}^{-1}(0) \text{ and } \pi(x) \geq 0\}$, as required. □

[2] i.e. locally modeled on the inclusion $H^k \subset H^n$ given by $(x_1, \ldots x_k) \mapsto (0, \ldots, 0, x_1, \ldots x_k)$.

3 Transversality

We continue to use the constant rank theorem to produce more manifolds, except now these will be cut out only *locally* by functions. Globally, they are cut out by intersecting with another submanifold. You should think that intersecting with a submanifold locally imposes a number of constraints equal to its codimension.

The problem is that the intersection of submanifolds need not be a submanifold; this is why the condition of transversality is so important - it guarantees that intersections are smooth.

Two subspaces $K, L \subset V$ of a vector space V are *transverse* when $K + L = V$, i.e. every vector in V may be written as a (possibly non-unique) linear combination of vectors in K and L. In this situation one can easily see that $\dim V = \dim K + \dim L - \dim K \cap L$, or equivalently

$$\mathrm{codim} V = \mathrm{codim} K + \mathrm{codim} L. \tag{52}$$

We may apply this to submanifolds as follows:

Definition 3.1. Let $K, L \subset M$ be regular submanifolds such that every point $p \in K \cap L$ satisfies

$$T_pK + T_pL = T_pM. \tag{53}$$

Then K, L are said to be *transverse* submanifolds and we write $K \pitchfork L$.

Proposition 3.2. *If $K, L \subset M$ are transverse submanifolds, then $K \cap L$ is either empty, or a submanifold of codimension* $\mathrm{codim} K + \mathrm{codim} L$.

Proof. Let $p \in K \cap L$. Then there is a neighbourhood U of p for which $K \cap U = f^{-1}(0)$ for 0 a regular value of a function $f : U \longrightarrow \mathbb{R}^{\mathrm{codim} K}$ and $L \cap U = g^{-1}(0)$ for 0 a regular value of a function $g : L \cap U \longrightarrow \mathbb{R}^{\mathrm{codim} L}$.

Then p must be a regular point for $(f, g) : L \cap M \cap U \longrightarrow \mathbb{R}^{\mathrm{codim} K + \mathrm{codim} L}$, since the kernel of its derivative is the intersection $\ker Df(p) \cap \ker Dg(p)$, which is exactly $T_pK \cap T_pL$, which has codimension $\mathrm{codim} K + \mathrm{codim} L$ by the transversality assumption, implying $D(f, g)(p)$ is surjective. Therefore $(f, g)|_{\tilde{U}}^{-1}(0, 0) = f^{-1}(0) \cap g^{-1}(0) = K \cap L \cap \tilde{U}$ is a submanifold. $\square$

Example 3.3 (Exotic spheres)**.** Consider the following intersections in $\mathbb{C}^5 \backslash 0$:

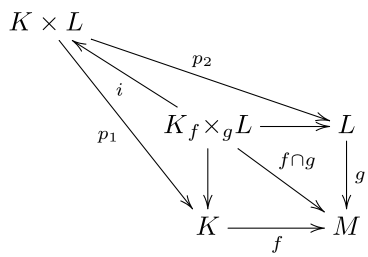
$$S_k^7 = \{z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} = 0\} \cap \{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 = 1\}. \tag{54}$$

This is a transverse intersection, and for $k = 1, \ldots, 28$ the intersection is a smooth manifold homeomorphic to S^7. These exotic 7-spheres were constructed by Brieskorn and represent each of the 28 diffeomorphism classes on S^7.

We may choose to phrase the previous transversality result in a slightly different way, in terms of the embedding maps k, l for K, L in M. Specifically, we say the maps k, l are transverse in the sense that $\forall a \in K, b \in L$ such that $k(a) = l(b) = p$, we have $\mathrm{im}(Dk(a)) + \mathrm{im}(Dl(b)) = T_pM$. The advantage of this approach is that it makes sense for any maps, not necessarily embeddings.

Definition 3.4. Two maps $f : K \longrightarrow M$, $g : L \longrightarrow M$ of manifolds are called *transverse* when $\mathrm{im}(Df(a)) + \mathrm{im}(Dg(b)) = T_pM$ for all a, b, p such that $f(a) = g(b) = p$.

Proposition 3.5. *If $f : K \longrightarrow M$, $g : L \longrightarrow M$ are transverse smooth maps, then $K_f\times_g L = \{(a, b) \in K \times L \ : \ f(a) = g(b)\}$ is naturally a smooth manifold equipped with commuting maps*



(55)

where i is the inclusion and $f \cap g : (a, b) \mapsto f(a) = g(b)$.

The manifold $K_f\times_g L$ of the previous proposition is called the *fiber product* of K with L over M, and is a generalization of the intersection product. It is often denoted simply by $K \times_M L$, when the maps to M are clear.

Proof. Consider the graphs $\Gamma_f \subset K \times M$ and $\Gamma_g \subset L \times M$. To impose $f(k) = g(l)$, we can take an intersection with the diagonal submanifold

$$\Delta = \{(k, m, l, m) \in K \times M \times L \times M\}. \tag{56}$$

Step 1. We show that the intersection $\Gamma = (\Gamma_f \times \Gamma_g) \cap \Delta$ is transverse. Let $f(k) = g(l) = m$ so that $x = (k, m, l, m) \in \Gamma$, and note that

$$T_x(\Gamma_f \times \Gamma_g) = \{((v, Df(v)), (w, Dg(w))), \ v \in T_kK, \quad w \in T_lL\} \tag{57}$$

whereas we also have

$$T_x(\Delta) = \{((v, m), (w, m)) \ : \ v \in T_kK, \quad w \in T_lL, \quad m \in T_pM\} \tag{58}$$

By transversality of f, g, any tangent vector $m_i \in T_pM$ may be written as $Df(v_i) + Dg(w_i)$ for some (v_i, w_i), $i = 1, 2$. In particular, we may decompose a general tangent vector to $M \times M$ as

$$(m_1, m_2) = (Df(v_2), Df(v_2)) + (Dg(w_1), Dg(w_1)) + (Df(v_1 - v_2), Dg(w_2 - w_1)), \tag{59}$$

leading directly to the transversality of the spaces (57), (58). This shows that Γ is a submanifold of $K \times M \times L \times M$.

Step 2. The projection map $\pi : K \times M \times L \times M \to K \times L$ takes Γ bijectively to $K_f\times_g L$. Since (57) is a graph, it follows that $\pi|_\Gamma : \Gamma \to K \times L$ is an injective immersion. Since the projection π is an open map, it also follows that $\pi|_\Gamma$ is a homeomorphism onto its image, hence is an embedding. This shows that $K_f\times_g L$ is a submanifold of $K \times L$. ∎

Example 3.6. If $K_1 = M \times Z_1$ and $K_2 = M \times Z_2$, we may view both K_i as "fibering" over M with fibers Z_i. If p_i are the projections to M, then $K_1 \times_M K_2 = M \times Z_1 \times Z_2$, hence the name "fiber product".

Example 3.7. Consider the Hopf map $p : S^3 \longrightarrow S^2$ given by composing the embedding $S^3 \subset \mathbb{C}^2 \backslash \{0\}$ with the projection $\pi : \mathbb{C}^2 \backslash \{0\} \longrightarrow \mathbb{C}P^1 \cong S^2$. Then for any point $q \in S^2$, $p^{-1}(q) \cong S^1$. Since p is a submersion, it is obviously transverse to itself, hence we may form the fiber product

$$S^3 \times_{S^2} S^3,$$

which is a smooth 4-manifold equipped with a map $p \cap p$ to S^2 with fibers $(p \cap p)^{-1}(q) \cong S^1 \times S^1$.

These are our first examples of nontrivial fiber bundles, which we shall explore later.

The following result is an exercise: just as we may take the product of a manifold with boundary K with a manifold without boundary L to obtain a manifold with boundary $K \times L$, we have a similar result for fiber products.

Proposition 3.8. *Let K be a manifold with boundary where L, M are without boundary. Assume that $f : K \longrightarrow M$ and $g : L \longrightarrow M$ are smooth maps such that both f and ∂f are transverse to g. Then the fiber product $K \times_M L$ is a manifold with boundary equal to $\partial K \times_M L$.*