There are five pages remaining in the notes, which I will be covering this week. The main topics are Stokes' theorem and the Mayer-Vietoris sequence. Please read ahead and complete the following problems.

Exercise 1. Consider S^n and its two stereographic coordinate charts φ_S, φ_N to $\mathbb{R}^n$. Using standard coordinates on $\mathbb{R}^n$, write down the coordinate expressions for a smooth, nowhere-vanishing n-form on S^n.

Exercise 2. Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$(r, \phi, \theta) \mapsto (r \sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi),$$

where (r, ϕ, θ) are standard Cartesian coordinates on $\mathbb{R}^3$.

- Compute $\varphi^* dx, \varphi^* dy, \varphi^* dz$ where (x, y, z) are Cartesian coordinates for $\mathbb{R}^3$.
- Compute $\varphi^*(dx \wedge dy \wedge dz)$.
- For any vector field X, define ι_X to be the unique degree -1 (i.e. it reduces degree by 1) derivation (i.e. $\iota_X(\alpha\wedge\beta) = \iota_X(\alpha)\wedge\beta + (-1)^{|\alpha|}\alpha\wedge\iota_X(\beta)$) of the algebra of differential forms such that $i_X(f) = 0$ and $i_X df = X(f)$ for $f \in \Omega^0(M)$. Compute the integral

  $$\int_{S_r^2} \iota_X(dx \wedge dy \wedge dz),$$

  for the vector field $X = \varphi_* \frac{\partial}{\partial r}$, where S_r^2 is the sphere of radius r.

Exercise 3. Use Stokes' theorem if necessary:

1. Let M be a compact orientable smooth n-manifold (without boundary) and let $\mu \in \Omega^{n-1}(M)$. Prove there exists a point $p \in M$ with $d\mu(p) = 0$.
2. For any sphere S^k, let $\iota : S^k \to \mathbb{R}^{k+1}$ be the usual inclusion, and let $v_k \in \Omega^k(S^k)$ be given by

   $$v_k = \iota^* \sum_{i=0}^{k} (-1)^i x^i dx^0 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^k.$$

   Show that v_k is closed and that $[v_k] \neq 0$ in the top de Rham cohomology group $H^k(S^k)$.

Exercise 4. Compute the de Rham cohomology groups (Using Mayer-Vietoris if necessary) of the following spaces, for all degrees. **Hand in only the second one:**

- $\mathbb{R}^3 - \{p\}$, for $p \in \mathbb{R}^3$ a point.
- $\mathbb{R}^3 - \{p_1 \cup p_2\}$ where p_i are distinct points?
- $\mathbb{R}^3 - \{\ell_1 \cup \ell_2\}$ where ℓ_i are non-intersecting lines?
- $\mathbb{R}^3 - \{\ell_1 \cup \ell_2\}$, assuming that l_1 intersects l_2 in exactly one point?

This question is a slightly easier version of the one John Nash asked in class in the movie "A Beautiful Mind".