Exercise 1. Let $f : M \to \mathbb{R}$ be a proper submersion. Then $V = \ker Tf$ defines a codimension 1 subbundle of TM called the vertical bundle.

1. Show, using a partition of unity, that it is possible to choose a rank 1 subbundle $H \subset TM$ complementary to V. Do not use a Riemannian metric.

2. Conclude that to any vector field v on $\mathbb{R}$ we may associate a unique vector field v^h on M which lies in H. This is called the horizontal lift of v.

3. Prove the preimages of any pair of points in the image of f are diffeomorphic manifolds.

Exercise 2. Consider the pair of vector fields $V = \partial_y$ and $W = y\partial_x - \partial_z$ on $\mathbb{R}^3$, where we use coordinates (x, y, z). Is it possible to find a 2-dimensional submanifold of $\mathbb{R}^3$ with the property that both V and W are tangent to it at all its points? If so, construct one; if not, why not?

Exercise 3. Let $\alpha = dz - xdy$ and $\beta = dx - wdy$ be 1-forms on $\mathbb{R}^4$ (or you can think of them as functions on $T\mathbb{R}^4$). Both α and β have 3-dimensional kernel on each tangent space.

1. Prove that $\ker \alpha \cap \ker \beta$ has dimension 2.

2. Give a local basis (V, W) for the above intersection.

3. Compute $[V, W]$. Is it linearly dependent on (V, W)?

4. Is it possible to find a 3-dimensional submanifold such that V, W, and $[V, W]$ are everwhere tangent to it? If so, construct one; if not, show why not.

Exercise 4. Let V be a finite dimensional vector space, and view it as a manifold M. Then $TM = V \times V$.

1. The trivial map $E : x \mapsto (x, x)$ defines a section of the tangent bundle, i.e. a vector field. Compute the time-t flow of this vector field and determine whether it is complete or not.

2. Suppose $A : V \to V$ is a linear map. Then the map $A : x \mapsto (x, Ax)$ defines a vector field on M; compute its flow and determine if it is complete.

3. If A, B are linear maps as above, compute the Lie derivative of the vector fields they determine. Verify the fact that if the vector fields commute, then the flows commute.