Exercise 1. Let K, L be submanifolds of a manifold M, and suppose that their intersection $K \cap L$ is also a submanifold. Then K, L are said to have *clean* intersection when, for each $p \in K \cap L$, we have $T_p(K \cap L) = T_pK \cap T_pL$. Show that there are coordinates near $p \in K \cap L$ such that K, L, and $K \cap L$ are given by linear subspaces of $\mathbb{R}^n$ of the form $V(x^{i_1}, \ldots, x^{i_k})$ for some subset of the coordinates. It is useful to use the algebraic geometry notation $V(x^1, \ldots, x^k)$ to mean the "vanishing" subspace $x^1 = \cdots = x^k = 0$.

Also, can the intersection of submanifolds be transverse but not clean? Can it be clean but not transverse? Give examples or proofs as necessary.

Exercise 2. Compute the mod 2 self-intersection number of the zero section $X \to TX$ for the manifolds $X \in \{S^1, S^2, \mathbb{R}P^2\}$, showing your reasoning. Deduce that every smooth vector field on $\mathbb{R}P^2$ must have a zero. Produce an explicit example of a vector field on $\mathbb{R}P^2$ with a single transverse zero.

Exercise 3. Let X be compact and $f : X \to Y$ smooth with $\dim X = \dim Y$ and Y connected. Recall that the mod 2 degree of f is defined in terms of the mod 2 intersection number as follows: $\deg_2(f) = I_2(f, \iota)$, where $\iota : y \mapsto Y$ is the inclusion map of a point $y \in Y$.

1. Prove that $\deg_2(f)$ is independent of the point $y \in Y$.
2. If Y is non-compact, prove that $\deg_2(f) = 0$.
3. A map $f : X \to Y$ is called *essential* when it is not homotopic to a constant map. Prove that if $\deg_2(f) = 1$, then f is essential.
4. Give example of a smooth surjective map $f : S^2 \to S^2$ with $\deg_2(f) = 0$.
5. Can there exist a smooth map $f : S^2 \to T^2$ with $\deg_2(f) = 1$? [Hint: consider two embedded circles C_1, C_2 in T^2 intersecting transversally at a single point.] Can there exist a smooth map of $\deg_2(f) = 1$ in the opposite direction? In each case, give proofs.

Exercise 4 (Jordan curve theorem)**.** Let $f : S^1 \to \mathbb{R}^2$ be an embedding and choose $p \in \mathbb{R}^2 \backslash f(S^1)$. Define $f_p : S^1 \to S^1$ by $f_p(z) = \frac{f(z)-p}{|f(z)-p|}$. Then we define the mod 2 winding number of f about p to be the degree of f_p, i.e. $w_2(f, p) = \deg_2(f_p)$. Warm up by computing $w_2(f, p)$ for the standard embedding of S^1 in $\mathbb{R}^2$, and for any p, with justifications.

1. Let $R_p(v)$ be the ray starting at p with direction $v \in S^1$. Prove that $v \in S^1$ is a critical value of f_p if and only if $R_p(v)$ is somewhere tangent to $f(S^1)$.
2. Show that $w_2(f, p)$ coincides with the number of points mod 2 in $R_p(v) \cap f(S^1)$, whenever v is a regular value of f_p.
3. Show that there are points $p, q \in \mathbb{R}^2 \backslash f(S^1)$ such that $w_2(f, p) = 0$ and $w_2(f, q) = 1$. Show that this implies that $\mathbb{R}^2 \backslash f(S^1)$ has at least two components.
4. Fix $a \in f(S^1)$. Show that it is possible to choose a coordinate chart (U, φ) containing a such that $\varphi(U)$ contains $(-2, 2) \times (-2, 2)$, $\varphi(a) = (0, 0)$, and $\varphi(U \cap f(S^1)) = \{(x, y) \ : \ y = 0\}$.
5. Prove that each point $p \in \mathbb{R}^2 \backslash f(S^1)$ may be connected by a continuous path to either $\varphi^{-1}(0, 1)$ or $\varphi^{-1}(0, -1)$. [Hint: recall the tubular neighbourhood theorem]. Conclude that $\mathbb{R}^2 \backslash f(S^1)$ has two connected components.