Exercise 1. Construct, using the stereographic charts for S^2 given in class, a smooth vector field on S^2 which vanishes exactly at 2 points, and another vector field which vanishes at exactly 1 point.

Exercise 2. Show that a compact manifold cannot have a smooth function without critical points.

Exercise 3.

1. Prove that the group $SO(3,\mathbb{R})$ of 3×3 real special orthogonal matrices, i.e. $SO(3,\mathbb{R}) = \{T \in SL(3,\mathbb{R}) \ : \ TT^\top = 1\}$ is a smooth submanifold of the vector space of 3×3 matrices.
2. Consider the subset of $T\mathbb{R}^3$ consisting of the vectors tangent to the 2-sphere $S^2 \subset \mathbb{R}^3$ and of unit length (we use the usual Euclidean length on $\mathbb{R}^3$, and the fact that $T\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$). Prove this subset is a submanifold.
3. Show that the intersection of the sphere $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$ in $\mathbb{C}^3$ with the complex cone $z_1^2 + z_2^2 + z_3^2 = 0$ is a submanifold.
4. Are any of the above three manifolds diffeomorphic to each other? Prove such assertions.
5. Are any of the above manifolds diffeomorphic to $\mathbb{R}P^3$? Prove your claim.

Bonus 3.1. Show that $\mathbb{R}P^3$ is null-cobordant.

Exercise 4. Prove that if K is a submanifold[1] of L and L is a submanifold of M, then K is a submanifold of M.

Exercise 5. Let K, K' be transverse submanifolds of codimension k, k' in the n-manifold M. Prove that each point $p \in K \cap K'$ has a neighbourhood $U \subset M$ and a diffeomorphism from U to a neighbourhood of the origin in $\mathbb{R}^n$ which takes K and K' to the coordinate planes $V(x_1, \ldots, x_k)$ and $V(x_{n-k'+1}, \ldots, x_n)$, respectively (Here V denotes the common zero set of its arguments).

Exercise 6. Let $f : M \to M$ be a smooth map and suppose p is a fixed point of f, i.e. $f(p) = p$. The point p is called a *Lefschetz fixed point* when the derivative map $f_* : T_pM \to T_pM$ does not have $+1$ as an eigenvalue.

Show that if M is compact and all fixed points for f are Lefschetz, then there are only finitely many fixed points for f.

Exercise 7. For any vector space V, we have a natural diffeomorphism $TV \cong V \times V$, where the projection $\pi_V : TV \to V$ corresponds to the first projection $\pi_1 : V \times V \to V$ given by $(a, b) \mapsto a$.

Let M be a smooth manifold, and choose a chart (U, φ) on M. By applying the tangent functor, and using the canonical isomorphism $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$, we obtain a chart $(TU, T\varphi)$ on the manifold TM. Repeating this procedure, we obtain a chart $(T(TU), T(T\varphi))$ on the manifold $T(TM)$.

Now define a diffeomorphism $J_U : T(TU) \to T(TU)$ by the composition $(T(T\varphi))^{-1} \circ j_U \circ T(T\varphi)$, where j_U is the automorphism of $(\varphi(U) \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n)$ given by

$$j_U : ((x, v), (u, w)) \mapsto ((x, u), (v, w)). \tag{1}$$

[1] We always mean *embedded* or *regular* submanifold when we say submanifold.

1. Show that for any atlas $\{(U_i, \varphi_i)\}$, we have $J_{U_i} = J_{U_j}$ on the overlap $T(T(U_i \cap U_j))$. Deduce that this defines a global diffeomorphism $J : T(TM) \to T(TM)$ and show that it is independent of the atlas used to construct it.

2. Consider the tangent bundle projection $\pi_M : TM \to M$. Applying the tangent functor, we obtain the smooth map $T\pi_M : T(TM) \to TM$. What is the relationship between this map and the tangent bundle projection $\pi_{TM} : T(TM) \to TM$?

3. Let $X : M \to TM$ be a vector field on M, and let its tangent mapping be $TX : TM \to T(TM)$. Is TX a vector field on the manifold TM?

4. Is $J \circ TX$ a vector field on the manifold TM?

Bonus 7.1. Show that J defines a natural transformation from the functor $T \circ T$ to itself, and that this natural transformation is an equivalence.