Exercise 1. Let Γ be a group, and give it the discrete topology. Suppose Γ acts continuously on the topological n-manifold M, meaning that the action map

$$\begin{aligned} \Gamma \times M &\xrightarrow{\rho} M \\ (h, x) &\longmapsto h \cdot x \end{aligned}$$

is continuous. Suppose also that the action is *free*, i.e. the stabilizer of each point is trivial. Finally, suppose the action is *properly discontinuous*, meaning that each $x \in M$ has a neighbourhood U such that $h \cdot U$ is disjoint from U for all nontrivial $h \in \Gamma$.

i) Show that the quotient map $\pi : M \to M/\Gamma$ is a local homeomorphism, where M/Γ is given the quotient topology. Conclude that M/Γ is locally homeomorphic to $\mathbb{R}^n$.

ii) Show that the quotient topology on M/Γ is uniquely determined by the requirement that π is a local homeomorphism.

iii) Show that π is an open map.

iv) Give an example where M/Γ is not Hausdorff.

Exercise 2. Let (Γ, M, ρ) be as in Exercise 1, and let $f : M \to N$ be a continuous map such that

$$f(h \cdot x) = f(x)$$

for all $x \in M$ and $h \in \Gamma$. Show that there is a unique map $\bar{f} : M/\Gamma \to N$ such that $\bar{f}(\pi(x)) = f(x)$ for all $x \in M$, and show that it is continuous.

Exercise 3. Let (Γ, M, ρ) be as in Exercise 1. Prove that M/Γ is Hausdorff if and only if the image of the map

$$\begin{aligned} \Gamma \times M &\longrightarrow M \times M \\ (g, x) &\longmapsto (gx, x) \end{aligned}$$

is closed in $M \times M$.

Exercise 4. Let $M = \mathbb{C}^n \setminus \{0\}$ and let the generator of $\Gamma = \mathbb{Z}$ act via $x \mapsto 2x$, for $x \in M$. Show that the quotient M/Γ is a manifold homeomorphic to $S^{2n-1} \times S^1$.

Exercise 5. Let $M = S^n$ and let $\Gamma = \mathbb{Z}_2$ act on M via $x \mapsto -x$. Show that M/Γ is homeomorphic to the projective space $\mathbb{R}P^n$, as it was defined in class.

Exercise 6. Consider the 3-sphere $S^3 \subset \mathbb{R}^4$. Using the isomorphism $\mathbb{R}^4 \cong \mathbb{C}^2$, we obtain the inclusion $\iota : S^3 \to \mathbb{C}^2 \setminus \{0\}$. Composing with the projection map $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$, we obtain

$$p = \pi \circ \iota : S^3 \to \mathbb{C}P^1,$$

known as the "Hopf fibration".

Using the coordinate charts given in class for S^3 and $\mathbb{C}P^1$, compute p in coordinates (one chart on each of the domain and codomain should suffice).