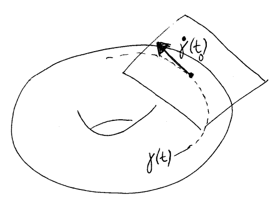
4

The Tangent Bundle

4.1 Tangent spaces

For embedded submanifolds $M \subseteq \mathbb{R}^n$, the tangent space T_pM at $p \in M$ can be defined as the set of all *velocity vectors* $v = \dot{\gamma}(0)$, where $\gamma\colon\ J \to M$ is a smooth curve with $\gamma(0) = p$; here $J \subseteq \mathbb{R}$ is an open interval around 0.



It turns out (not entirely obvious!) that T_pM becomes a vector subspace of $\mathbb{R}^n$. (Warning: In pictures we tend to draw the tangent space as an *affine* subspace, where the origin has been moved to p.)

Example 4.1. Consider the sphere $S^n \subseteq \mathbb{R}^{n+1}$, given as the set of x such that $||x||^2 = 1$. A curve $\gamma(t)$ lies in S^n if and only if $||\gamma(t)|| = 1$. Taking the derivative of the equation $\gamma(t) \cdot \gamma(t) = 1$ at $t = 0$, we obtain (after dividing by 2, and using $\gamma(0) = p$)

$$p \cdot \dot{\gamma}(0) = 0.$$

That is, T_pM consists of vectors $v \in \mathbb{R}^{n+1}$ that are orthogonal to $p \in \mathbb{R}^3 \backslash \{0\}$. It is not hard to see that every such vector v is of the form $\dot{\gamma}(0)$,* hence that

$$T_pS^n = (\mathbb{R}p)^\perp,$$

the hyperplane orthogonal to the line through p.

* Given v, take $\gamma(t) = (p+tv)/||p+tv||$.

Exercise 45. What is $T_0\mathbb{R}^2$? What is $T_0\mathbb{R}^n$? How about $T_p\mathbb{R}^n$ for some other $p \in \mathbb{R}^n$?

To extend this idea to general manifolds, note that the vector $v = \dot{\gamma}(0)$ defines a "directional derivative" $C^\infty(M) \to \mathbb{R}$:

$$v : f \mapsto \tfrac{d}{dt}|_{t=0} f(\gamma(t)).$$

Exercise 46. Fix some $p \in \mathbb{R}^n$ Show that the directional derivative in $\mathbb{R}^n$

$$v(f) = \lim_{t\to 0} \frac{f(p+tv) - f(p)}{t}$$

(for $v \in T_p\mathbb{R}^n \cong \mathbb{R}^n$ and $f \in C^\infty(\mathbb{R}^n)$) has the following properties. For any $v \in T_pM$ and $f, g \in C^\infty(\mathbb{R}^n)$:

a) $v : C^\infty \to \mathbb{R}$ is well-defined (that is, the limit exists for any $f \in C^\infty$).
b) $v : C^\infty \to \mathbb{R}$ is linear: $v(\lambda f + \mu g) = \lambda v(f) + \mu v(g)$.
c) $v : C^\infty \to \mathbb{R}$ is a *derivation*: $v(fg) = v(f)g + fv(g)$.

For a general manifold, we will define T_pM as a set of directional derivatives.

Definition 4.1 (Tangent spaces – first definition). *Let M be a manifold, $p \in M$. The tangent space T_pM is the set of all linear maps $v : C^\infty(M) \to \mathbb{R}$ of the form*

$$v(f) = \tfrac{d}{dt}|_{t=0} f(\gamma(t))$$

for some smooth curve $\gamma \in C^\infty(J, M)$ with $\gamma(0) = p$.

The elements $v \in T_pM$ are called the *tangent vectors* to M at p.

The following local coordinate description makes it clear that T_pM is a linear subspace of the vector space $L(C^\infty(M), \mathbb{R})$ of linear maps $C^\infty(M) \to \mathbb{R}$, of dimension equal to the dimension of M.

Theorem 4.1. *Let (U, φ) be a coordinate chart around p. A linear map $v : C^\infty(M) \to \mathbb{R}$ is in T_pM if and only if it has the form,*

$$v(f) = \sum_{i=1}^m a^i \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}\Big|_{u=\varphi(p)}$$

for some $a = (a^1, \ldots, a^m) \in \mathbb{R}^m$.

Proof. Given a linear map v of this form, let $\tilde{\gamma} : \mathbb{R} \to \varphi(U)$ be a curve with $\tilde{\gamma}(t) = \varphi(p) + ta$ for $|t|$ sufficiently small. Let $\gamma = \varphi^{-1} \circ \tilde{\gamma}$. Then

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} f(\gamma(t)) &= \frac{d}{dt}\Big|_{t=0} (f \circ \varphi^{-1})(\varphi(p) + ta) \\ &= \sum_{i=1}^m a^i \, \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}\Big|_{u=\varphi(p)}, \end{aligned}$$

by the chain rule. Conversely, given any curve γ with $\gamma(0) = p$, let $\tilde{\gamma} = \varphi \circ \gamma$ be the corresponding curve in $\varphi(U)$ (defined for small $|t|$). Then $\widetilde{\gamma}(0) = \varphi(p)$, and

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} f(\gamma(t)) &= \frac{d}{dt}\Big|_{t=0} (f \circ \varphi^{-1})(\tilde{\gamma}(t)) \\ &= \sum_{i=1}^{m} a^i \, \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}|_{u=\gamma(p)}, \end{aligned}$$

where $a = \frac{d\tilde{\gamma}}{dt}\Big|_{t=0}$. □

We can use this result as an alternative definition of the tangent space, namely:

Definition 4.2 (Tangent spaces – second definition). *Let (U, φ) be a chart around p. The tangent space T_pM is the set of all linear maps $v : C^\infty(M) \to \mathbb{R}$ of the form*

$$v(f) = \sum_{i=1}^{m} a^i \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}\Big|_{u=\varphi(p)} \tag{4.1}$$

for some $a = (a^1, \ldots, a^m) \in \mathbb{R}^m$.

Remark 4.1. From this version of the definition, it is immediate that T_pM is an m-dimensional vector space. It is not immediately obvious from this second definition that T_pM is independent of the choice of coordinate chart, but this follows from the equivalence with the first definition. Alternatively, one may check directly that the subspace of $L(C^\infty(M), \mathbb{R})$ characterized by (4.1) does not depend on the chart, by studying the effect of a change of coordinates.

According to (4.1), any choice of coordinate chart (U, φ) around p defines a vector space isomorphism $T_pM \cong \mathbb{R}^m$, taking v to $a = (a^1, \ldots, a^m)$. In particular, we see that if $U \subseteq \mathbb{R}^m$ is an open subset, and $p \in U$, then T_pU is the subspace of the space of linear maps $C^\infty(M) \to \mathbb{R}$ spanned by the partial derivatives at p. That is, T_pU has a basis

$$\frac{\partial}{\partial x^1}|_p, \ldots, \frac{\partial}{\partial x^m}|_p$$

identifying $T_pU \equiv \mathbb{R}^m$. Given

$$v = \sum a^i \frac{\partial}{\partial x^i}|_p$$

the coefficients a^i are obtained by applying v to the coordinate functions $x^1, \ldots, x^m : U \to \mathbb{R}$, that is, $a^i = v(x^i)$.

We now describe yet another approach to tangent spaces which again characterizes "directional derivatives" in a coordinate-free way, but without reference to curves γ. Note first that every tangent vector satisfies the *product rule*, also called the *Leibniz rule*:

Lemma 4.1. *Let $v \in T_pM$ be a tangent vector at $p \in M$. Then*

$$v(fg) = f(p)\, v(g) + v(f)\, g(p) \tag{4.2}$$

for all $f, g \in C^\infty(M)$.

Proof. Letting v be represented by a curve γ, this follows from

$$\frac{d}{dt}\Big|_{t=0}\Big(f\big(\gamma(t)\big)g\big(\gamma(t)\big)\Big) = f(p)\Big(\frac{d}{dt}\Big|_{t=0}g\big(\gamma(t)\big)\Big) + \Big(\frac{d}{dt}\Big|_{t=0}f\big(\gamma(t)\big)\Big)g(p).$$

□

Alternatively, in local coordinates it is just the product rule for partial derivatives. It turns out that the product rule completely characterizes tangent vectors:

Exercise 47. Suppose that $v : C^\infty(M) \to \mathbb{R}$ is a linear map satisfying the product rule (4.2). Prove the following two facts, which will be used in the proof of Theorem 4.2 below:

a) v vanishes on constants. That is, if $f \in C^\infty(M)$ is the constant map, then $v(f) = 0$.
b) Suppose $f, g \in C^\infty(M)$ with $f(p) = g(p) = 0$. Then $v(fg) = 0$.

Theorem 4.2. *A linear map $v : C^\infty(M) \to \mathbb{R}$ defines an element of T_pM if and only if it satisfies the product rule (4.2).*

The proof of this result will require the following fact from multivariable calculus:

Lemma 4.2 (Hadamard Lemma). *Let $U = B_R(0) \subseteq \mathbb{R}^m$ be an open ball of radius $R > 0$ and $h \in C^\infty(U)$ a smooth function. Then there exist smooth functions $h_i \in C^\infty(U)$ with*

$$h(u) = h(0) + \sum_{i=1}^{m} u^i h_i(u)$$

for all $u \in U$. Here $h_i(0) = \frac{\partial h}{\partial u^i}(0)$.

Proof. Let h_i be the functions defined for $u = (u^1, \ldots, u^m) \in U$ by

$$h_i(u) = \begin{cases} \frac{1}{u^i}\big(h(u^1,\ldots,u^i,0,\ldots,0) - h(u^1,\ldots,u^{i-1},0,0,\ldots,0)\big) & \text{if } \; u^i \neq 0 \\ \frac{\partial h}{\partial u^i}(u^1,\ldots,u^{i-1},0,0,\ldots,0) & \text{if } \; u^i = 0 \end{cases}$$

Using Taylor's formula with remainder, one sees that these functions are smooth†. If all $u^i \neq 0$, then the sum $\sum_{i=1}^m u^i h_i(u)$ is a telescoping sum, equal to $h(u) - h(0)$. By continuity, this result extends to all u. Finally, evaluating the derivative

$$\frac{\partial h}{\partial u^i} = \frac{\partial}{\partial u^i}\left(h(0) + \sum_{i=1}^{m} u^i h_i(u)\right) = h_i(u) + \sum_k u^k \frac{\partial h_k}{\partial u^i}$$

at $u = 0$, we see that $\frac{\partial h}{\partial u^i}\big|_{u=0} = h_i(0)$. □

† It is a well-known fact from calculus (proved e.g. by using Taylor's theorem with remainder) that if f is a smooth function of a real variable x, then the function g, defined as $g(x) = x^{-1}(f(x) - f(0))$ for $x \neq 0$ and $g(0) = f'(0)$, is smooth.

Proof (Theorem 4.2). Let $v : C^\infty(M) \to \mathbb{R}$ be a linear map satisfying the product rule (4.2).

Step 1: *If $f_1 = f_2$ on some open neighborhood U of p, then $v(f_1) = v(f_2)$.*

Equivalently, letting $f = f_1 - f_2$, we show that $v(f) = 0$ if $f = 0$ on U. Choose a 'bump function' $\chi \in C^\infty(M)$ with $\chi(p) = 1$, with $\chi|_{M\setminus U} = 0$. Then $f\chi = 0$. The product rule tells us that

$$0 = v(f\chi) = v(f)\chi(p) + v(\chi)f(p) = v(f).$$

Step 2: *Let (U, φ) be a chart around p, with image $\widetilde{U} = \varphi(U)$. Then there is unique linear map $\widetilde{v} :\ C^\infty(\widetilde{U}) \to \mathbb{R}$ such that $\widetilde{v}(\widetilde{f}) = v(f)$ whenever $\widetilde{f}$ agrees with $f \circ \varphi^{-1}$ on some neighborhood of $\widetilde{p}$.*

Given $\widetilde{f}$, we can always find a function f such that $\widetilde{f}$ agrees with $f \circ \varphi^{-1}$ on some neighborhood of $\widetilde{p}$. Given another such function g, it follows from Step 1 that $v(f) = v(g)$.

Step 3: *In a chart (U, φ) around p, the map $v :\ C^\infty(M) \to \mathbb{R}$ is of the form (4.1).*

Since the condition (4.1) does not depend on the choice of chart around p, we may assume that $\widetilde{p} = \varphi(p) = 0$, and that $\widetilde{U}$ is an open ball of some radius $R > 0$ around 0. Define $\widetilde{v}$ as in Step 2. Since v satisfies the product rule on $C^\infty(M)$, the map $\widetilde{v}$ satisfies the product rule on $C^\infty(\widetilde{U})$. Given $f \in C^\infty(M)$, consider the Taylor expansion of the coordinate expression $\tilde{f} = f \circ \varphi^{-1}$ near $u = 0$:

$$\tilde{f}(u) = \widetilde{f}(0) + \sum_i u^i \frac{\partial \tilde{f}}{\partial u^i}\Big|_{u=0} + \tilde{r}(u)$$

The remainder term $\tilde{r}$ is a smooth function that vanishes at $u = 0$ together with its first derivatives. By Lemma 4.2, it can be written in the form $\tilde{r}(u) = \sum_i u^i \tilde{r}_i(u)$ where $\tilde{r}_i$ are smooth functions that vanish at 0. Let us now apply $\widetilde{v}$ to the formula for $\widetilde{f}$. Since $\widetilde{v}$ vanishes on products of functions vanishing at 0 (by Exercise 4.1), we have that $\widetilde{v}(\widetilde{r}) = 0$. Since it also vanishes on constants, we obtain

$$v(f) = \widetilde{v}(\widetilde{f}) = \sum_i a^i \frac{\partial \tilde{f}}{\partial u^i}\Big|_{u=0},$$

where we put $a^i = \widetilde{v}(u^i)$.

To summarize, we have the following alternative definition of tangent spaces:

Definition 4.3 (Tangent spaces – third definition). *The tangent space T_pM is the space of linear maps $C^\infty(M) \to \mathbb{R}$ satisfying the product rule,*

$$v(fg) = f(p)v(g) + v(f)\,g(p)$$

for all $f, g \in C^\infty(M)$.

At first sight, this characterization may seem a bit less intuitive then the definition as directional derivatives along curves. But it has the advantage of being less redundant – a tangent vector may be represented by many curves. Also, as in the coordinate definition it is immediate that T_pM is a linear subspace of the vector space $L(C^\infty(M),\mathbb{R})$. One may still want to use local charts, however, to prove that this vector subspace has dimension equal to the dimension of M.

The following remark gives yet another characterization of the tangent space. Please read it only if you like it abstract – *otherwise skip this*!

Remark 4.2 (A fourth definition). There is a fourth definition of T_pM, as follows. For any $p \in M$, let $C_p^\infty(M)$ denotes the subspace of functions vanishing at p, and let $C_p^\infty(M)^2$ consist of finite sums $\sum_i f_i g_i$ where $f_i, g_i \in C_p^\infty(M)$. We have a direct sum decomposition

$$C^\infty(M) = \mathbb{R} \oplus C_p^\infty(M),$$

where $\mathbb{R}$ is regarded as the constant functions. Since any tangent vector $v\colon C^\infty(M) \to \mathbb{R}$ vanishes on constants, v is effectively a map $v : C_p^\infty(M) \to \mathbb{R}$. By the product rule, v vanishes on the subspace $C_p^\infty(M)^2 \subseteq C_p^\infty(M)$. Thus v descends to a linear map $C_p^\infty(M)/C_p^\infty(M)^2 \to \mathbb{R}$, i.e. an element of the dual space $(C_p^\infty(M)/C_p^\infty(M)^2)^*$. The map

$$T_pM \to (C_p^\infty(M)/C_p^\infty(M)^2)^*$$

just defined is an *isomorphism*, and can therefore be used as a definition of T_pM. This may appear very fancy on first sight, but really just says that a tangent vector is a linear functional on $C^\infty(M)$ that vanishes on constants and depends only on the first order Taylor expansion of the function at p. Furthermore, this viewpoint lends itself to generalizations which are relevant to algebraic geometry and non-commutative geometry: The 'vanishing ideals' $C_p^\infty(M)$ are the maximal ideals in the algebra of smooth functions, with $C_p^\infty(M)^2$ their second power (in the sense of products of ideals). Thus, for any maximal ideal $\mathscr{I}$ in a commutative algebra $\mathscr{A}$ one may regard $(\mathscr{I}/\mathscr{I}^2)^*$ as a 'tangent space'.

After this lengthy discussion of tangent spaces, observe that the velocity vectors of curves are naturally elements of the tangent space. Indeed, let $J \subseteq \mathbb{R}$ be an open interval, and $\gamma \in C^\infty(J,M)$ a smooth curve. Then for any $t_0 \in J$, the tangent (or *velocity*) vector

$$\dot{\gamma}(t_0) \in T_{\gamma(t_0)}M.$$

at time t_0 is given in terms of its action on functions by

$$(\dot{\gamma}(t_0))(f) = \frac{d}{dt}\Big|_{t=t_0} f(\gamma(t))$$

We will also use the notation $\frac{d\gamma}{dt}(t_0)$ or $\frac{d\gamma}{dt}|_{t_0}$ to denote the velocity vector.

4.2 The Tangent Map

4.2.1 Definition of the tangent map, basic properties

For smooth maps $F \in C^\infty(U,V)$ between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ of Euclidean spaces, and any given $p \in U$, we considered the derivative to be the linear map

$$D_pF : \mathbb{R}^m \to \mathbb{R}^n,\ a \mapsto \frac{d}{dt}\Big|_{t=0} F(p+ta).$$

The following definition generalizes the derivative to smooth maps between manifolds.

Definition 4.4. *Let M,N be manifolds and $F \in C^\infty(M,N)$. For any $p \in M$, we define the* tangent map *to be the linear map*

$$T_pF : T_pM \to T_{F(p)}N$$

given by

$$\big(T_pF(v)\big)(g) = v(g \circ F)$$

for $v \in T_pM$ and $g \in C^\infty(N)$.

We leave it as an exercise to check that the right hand side does indeed define a tangent vector:

Exercise 48. Show that for all $v \in T_pM$, the map $g \mapsto v(g \circ F)$ satisfies the product rule at $q = F(p)$, hence defines an element of T_qN.

Proposition 4.1. *If $v \in T_pM$ is represented by a curve $\gamma : J \to M$, then $(T_pF)(v)$ is represented by the curve $F \circ \gamma$.*

Exercise 49. Prove Proposition 4.1.

Remark 4.3 (Pull-backs, push-forwards). For smooth maps $F \in C^\infty(M,N)$, one can consider various 'pull-backs' of objects on N to objects on M, and 'push-forwards' of objects on M to objects on N. Pull-backs are generally denoted by F^*, push-forwards by F_*. For example, functions on N pull back

$$g \in C^\infty(N) \ \rightsquigarrow\ F^*g = g \circ F \in C^\infty(M).$$

Curves on M push forward:

$$\gamma : J \to M \ \rightsquigarrow\ F_*\gamma = F \circ \gamma : J \to N.$$

Tangent vectors to M also push forward,

$$v \in T_pM \ \rightsquigarrow\ F_*(v) = (T_pF)(v).$$

The definition of the tangent map can be phrased in these terms as $(F_*v)(g) = v(F^*g)$. Note also that if v is represented by the curve γ, then F_*v is represented by the curve $F_*\gamma$.

Proposition 4.2 (Chain rule). *Let M,N,Q be manifolds. Under composition of maps $F \in C^\infty(M,N)$ and $F' \in C^\infty(N,Q)$,*

$$T_p(F' \circ F) = T_{F(p)}F' \circ T_pF.$$

Exercise 50. Prove Proposition 4.2. Give two different proofs: one using the action of tangent vectors on functions, and one using Proposition 4.1.

Exercise 51. a) Show that the tangent map of the identity map $\mathrm{id}_M : M \to M$ at $p \in M$ is the identity map on the tangent space:

$$T_p \,\mathrm{id}_M = \mathrm{id}_{T_pM}.$$

b) Show that if $F \in C^\infty(M,N)$ is a diffeomorphism, then T_pF is a linear isomorphism, with inverse

$$(T_pF)^{-1} = (T_{F(p)}F^{-1}).$$

c) Suppose that $F \in C^\infty(M,N)$ is a *constant map*, that is, $F(M) = \{q\}$ for some element $q \in N$. Show that $T_pF = 0$ for all $p \in M$.

4.2.2 Coordinate description of the tangent map

To get a better understanding of the tangent map, let us first consider the special case where $F \in C^\infty(U,V)$ is a smooth map between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. For $p \in U$, the tangent space T_pU is canonically identified with $\mathbb{R}^m$, using the basis

$$\frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^m}\Big|_p \in T_pU$$

of the tangent space (cf. Remark 4.1). Similarly, $T_{F(p)}V \cong \mathbb{R}^n$, using the basis given by partial derivatives $\frac{\partial}{\partial y^j}|_{F(p)}$. Using this identifications, the tangent map becomes a linear map $T_pF : \mathbb{R}^m \to \mathbb{R}^n$, i.e. it is given by an $n \times m$-matrix. This matrix is exactly the Jacobian:

Proposition 4.3. *Let $F \in C^\infty(U,V)$ be a smooth map between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. For all $p \in M$, the tangent map T_pF is just the derivative (i.e., Jacobian matrix) D_pF of F at p.*

Proof. For $g \in C^\infty(V)$, we calculate

$$\begin{aligned}\Big((T_pF)\big(\frac{\partial}{\partial x^i}\Big|_p\big)\Big)(g) &= \frac{\partial}{\partial x^i}\Big|_p (g \circ F)\\ &= \sum_{j=1}^n \frac{\partial g}{\partial y^j}\Big|_{F(p)} \frac{\partial F^j}{\partial x^i}\Big|_p\\ &= \Big(\sum_{j=1}^n \frac{\partial F^j}{\partial x^i}\Big|_p \frac{\partial}{\partial y^j}\Big|_{F(p)}\Big)(g).\end{aligned}$$

This shows

$$(T_pF)\Big(\frac{\partial}{\partial x^i}\Big|_p\Big) = \sum_{j=1}^{n} \frac{\partial F^j}{\partial x^i}\Big|_p \frac{\partial}{\partial y^j}\Big|_{F(p)}.$$

Hence, in terms of the given bases of T_pU and $T_{F(p)}V$, the matrix of the linear map T_pF has entries $\frac{\partial F^j}{\partial x^i}\Big|_p$.

Remark 4.4. For $F \in C^\infty(U,V)$, it is common to write $y = F(x)$, and accordingly write $(\frac{\partial y^j}{\partial x^i})_{i,j}$ for the Jacobian. In these terms, the derivative reads as

$$T_pF\Big(\frac{\partial}{\partial x^i}\Big|_p\Big) = \sum_j \frac{\partial y^j}{\partial x^i}\Big|_p \frac{\partial}{\partial y^j}\Big|_{F(p)}.$$

This formula is often used for explicit calculations.

Exercise 52. Consider $\mathbb{R}^2$ with coordinates x, y. Introduce a new coordinate system, polar coordinates r, θ given by

$$x = r\cos\theta \ , \ y = r\sin\theta$$

on the open set $(r,\theta) \in (0,\infty) \times (-\pi,\pi)$. Express the tangent vectors

$$\frac{\partial}{\partial r}\Big|_p \ , \ \frac{\partial}{\partial \theta}\Big|_p$$

as a combination of the tangent vectors

$$\frac{\partial}{\partial x}\Big|_p \ , \ \frac{\partial}{\partial y}\Big|_p$$

(with coefficients are C^∞ functions of x, y).

For a general smooth map $F \in C^\infty(M,N)$, we obtain a similar description once we pick coordinate charts. Given $p \in M$, choose charts (U,φ) around p and (V,ψ) around $F(p)$, with $F(U) \subseteq V$. Let $\widetilde{U} = \varphi(U)$, $\widetilde{V} = \psi(V)$, and put

$$\widetilde{F} = \psi \circ F \circ \varphi^{-1} : \ \widetilde{U} \to \widetilde{V}.$$

Since the coordinate map $\varphi : \ U \to \mathbb{R}^m$ is a diffeomorphism onto $\widetilde{U}$, It gives an isomorphism (cf. Exercise 51)

$$T_p\varphi : \ T_pU \to T_{\varphi(p)}\widetilde{U} = \mathbb{R}^m.$$

Similarly, $T_{F(p)}\psi$ gives an isomorphism of $T_{F(p)}V$ with $\mathbb{R}^n$. Note also that since $U \subseteq M$ is open, we have that $T_pU = T_pM$. We obtain,

$$T_{\varphi(p)}\widetilde{F} = T_{F(p)}\psi \circ T_pF \circ (T_p\varphi)^{-1}.$$

which may be depicted in a commutative diagram

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{D_{\varphi(p)}\widetilde{F}} & \mathbb{R}^n \\ {\scriptstyle T_p\varphi}\uparrow\cong & & \cong\uparrow{\scriptstyle T_{F(p)}\psi} \\ T_pM = T_pU & \xrightarrow[T_pF]{} & T_{F(p)}V = T_{F(p)}N \end{array}$$

Now that we have recognized T_pF *as the derivative expressed in a coordinate-free way*, we may liberate some of our earlier definitions from coordinates:

Definition 4.5. *Let $F \in C^\infty(M,N)$.*

- *The* rank of F at $p \in M$*, denoted* $\text{rank}_p(F)$*, is the rank of the linear map* T_pF*.*
- F *has* maximal rank at p *if* $\text{rank}_p(F) = \min(\dim M, \dim N)$.
- F *is a* submersion *if* T_pF *is surjective for all* $p \in M$*,*
- F *is an* immersion *if* T_pF *is injective for all* $p \in M$*,*
- F *is a* local diffeomorphism *if* T_pF *is an isomorphism for all* $p \in M$*.*
- $p \in M$ *is a* critical point *of* F *is* T_pF *does not have maximal rank at* p*.*
- $q \in N$ *is a* regular value *of* F *if* T_pF *is surjective for all* $p \in F^{-1}(q)$ *(in particular, if* $q \notin F(M)$*).*
- $q \in N$ *is a* singular value *(sometimes called* critical value*) if it is not a regular value.*

Exercise 53. As an example of the advantage of an intrinsic (i.e. coordinate free) definition, use this new definitions to show that the compositions of two submersions is again a submersion, and that the composition of two immersions is an immersion.

4.2.3 Tangent spaces of submanifolds

Suppose $S \subseteq M$ is a submanifold, and $p \in S$. Then the tangent space T_pS is canonically identified as a subspace of T_pM. Indeed, since the inclusion $i:\ S \hookrightarrow M$ is an immersion, the tangent map is an injective linear map,

$$T_pi:\ T_pS \to T_pM,$$

and we identify T_pS with the subspace given as the image of this map[‡]. As a special case, we see that whenever M is realized as a submanifold of $\mathbb{R}^n$, then its tangent spaces T_pM may be viewed as subspaces of $T_p\mathbb{R}^n = \mathbb{R}^n$.

Proposition 4.4. *Let $F \in C^\infty(M,N)$ be a smooth map, having $q \in N$ as a regular value, and let $S = F^{-1}(q)$. For all $p \in S$,*

$$T_pS = \ker(T_pF),$$

as subspaces of T_pM.

[‡] Hopefully, the identifications are not getting too confusing: S gets identified with $i(S) \subseteq M$, hence also $p \in S$ with its image $i(p)$ in M, and T_pS gets identified with $(T_pi)(T_pS) \subseteq T_pM$.

Proof. Let $m = \dim M$, $n = \dim N$. Since T_pF is surjective, its kernel has dimension $m - n$. By the normal form for submersions, this is also the dimension of S, hence of T_pS. It is therefore enough to show that $T_pS \subseteq \ker(T_pF)$. Letting $i: S \to M$ be he inclusion, we have to show that

$$T_pF \circ T_pi = T_p(F \circ i)$$

is the zero map. But $F \circ i$ is a *constant map*, taking all points of S to the constant value $q \in N$. The tangent map to a constant map is just zero (Exercise 51). Hence $T_p(F \circ i) = 0$. □

As a special case, we can describe the tangent spaces to level sets:

Corollary 4.1. *Suppose $V \subseteq \mathbb{R}^n$ is open, and $q \in \mathbb{R}^k$ is a regular value of $F \in C^\infty(M, \mathbb{R}^k)$, defining an embedded submanifold $M = F^{-1}(q)$. For all $p \in M$, the tangent space $T_pM \subseteq T_p\mathbb{R}^n = \mathbb{R}^n$ is given as*

$$T_pM = \ker(T_pF) \equiv \ker(D_pF).$$

Example 4.2. Recall that at the beginning of the chapter we have calculated T_pS^n directly from the curves definition of the tangent space. Here is another way of doing so. We Let $F: \mathbb{R}^{n+1} \to \mathbb{R}$ be the map $F(x) = x \cdot x = (x^0)^2 + \cdots + (x^n)^2$. Then, for all $p \in F^{-1}(1) = S^n$,

$$(D_pF)(a) = \frac{d}{dt}\Big|_{t=0} F(p + ta) = \frac{d}{dt}\Big|_{t=0} (p + ta) \cdot (p + ta) = 2p \cdot a,$$

hence

$$T_pS^n = \{a \in \mathbb{R}^{n+1} |\ a \cdot p = 0\} = \operatorname{span}(p)^\perp.$$

As another typical application, suppose that $S \subseteq M$ is a submanifold, and $f \in C^\infty(S)$ is a smooth function given as the restriction $f = h|_S$ of a smooth function $h \in C^\infty(M)$. Consider the problem of finding the critical points $p \in S$ of f, that is,

$$\operatorname{Crit}(f) = \{p \in S |\ T_pf = 0\}.$$

Letting $i: S \to M$ be the inclusion, we have $f = h|_S = h \circ i$, hence $T_pf = T_ph \circ T_pi$. It follows that $T_pf = 0$ if and only if T_ph vanishes on the range of T_pi, that is on T_pS:

$$\operatorname{Crit}(f) = \{p \in S |\ T_pS \subseteq \ker(T_ph)\}.$$

If $M = \mathbb{R}^m$, then T_ph is just the Jacobian D_ph, whose kernel is sometimes rather easy to compute – in any case this approach tends to be much faster than a calculation in charts.

Exercise 54. Find the critical points of

$$f: S^2 \to \mathbb{R}, \ f(x, y, z) = xy.$$

(a) Write $f = h \circ i$ for the appropriate map h. Then find $T_ph = D_ph$.
(b) Examine what happens when $D_ph = 0$.

(c) Suppose $D_p h \neq 0$. Find $\ker T_p h$, and find when does $T_p S^2 \subseteq \ker(T_p h)$.
(d) Summarize your finding by listing the critical points (you should have found six of them).

Exercise 55. Let $S \subseteq \mathbb{R}^3$ be a surface. Show that $p \in S$ is a critical point of the function $f \in C^\infty(S)$ given by $f(x,y,z) = z$, if and only if $T_p S$ is the xy-plane.

Exercise 56. Show that the equations

$$x^2 + y = 0 \ , \ x^2 + y^2 + z^3 + w^4 + y = 1$$

define a two dimensional submanifold S of $\mathbb{R}^4$, and find the equation of the tangent space at the point $(x_0, y_0, z_0, w_0) = (-1,-1,-1,-1)$.

(a) Reduce the problem to showing that $(0,1)$ is a regular value of a certain function $F \in C^\infty(\mathbb{R}^4, \mathbb{R}^2)$.
(b) Compute $T_p F = D_p F$, and find the critical points of F. Show that $(0,1)$ is a regular value.
(c) Compute $D_p F$ at $(-1,-1,-1,-1)$ and use Corollary 4.1 to find the equation of the tangent space.

Example 4.3. We had discussed various *matrix Lie groups* G as examples of manifolds. By definition, these are submanifolds $G \subseteq \mathrm{Mat}_{\mathbb{R}}(n)$, consisting of invertible matrices with the properties

$$A, B \in G \Rightarrow AB \in G, \ \ A \in G \Rightarrow A^{-1} \in G.$$

The tangent space to the identity (group unit) for such matrix Lie groups G turns out to be important; it is commonly denoted by lower case Fraktur letters:

$$\mathfrak{g} = T_I G.$$

Some concrete examples:

1. The matrix Lie group

$$\mathrm{GL}(n,\mathbb{R}) = \{A \in \mathrm{Mat}_{\mathbb{R}}(n) |\ \det(A) \neq 0\}$$

of *all* invertible matrices is an open subset of $\mathrm{Mat}_{\mathbb{R}}(n)$, hence

$$\mathfrak{gl}(n,\mathbb{R}) = \mathrm{Mat}_{\mathbb{R}}(n)$$

is the entire space of matrices.

2. For the group $\mathrm{O}(n)$, consisting of matrices with $F(A) := A^\top A = I$, we have computed $T_A F(X) = X^\top A + AX^\top$. For $A = I$, the kernel of this map is

$$\mathfrak{o}(n) = \{X \in \mathrm{Mat}_\mathbb{R}(n) |\; X^\top = -X\}.$$

3. For the group $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{Mat}_\mathbb{R}(n) |\; \det(A) = 1\}$, given as the level set $F^{-1}(1)$ of the function $\det:\ \mathrm{Mat}_\mathbb{R}(n) \to \mathbb{R}$, we calculate

$$D_A F(X) = \frac{d}{dt}\Big|_{t=0} F(A+tX) = \frac{d}{dt}\Big|_{t=0} \det(A+tX) = \frac{d}{dt}\Big|_{t=0} \det(I+tA^{-1}X) = \mathrm{tr}(A^{-1}X),$$

where $\mathrm{tr}:\ \mathrm{Mat}_\mathbb{R}(n) \to \mathbb{R}$ is the trace (sum of diagonal entries). (See exercise below.) Hence

$$\mathfrak{sl}(n,\mathbb{R}) = \{X \in \mathrm{Mat}_\mathbb{R}(n) |\; \mathrm{tr}(X) = 0\}.$$

Exercise 57. Show that for every $X \in \mathrm{Mat}_\mathbb{R}(n)$,

$$\frac{d}{dt}\Big|_{t=0} \det(I+tX) = \mathrm{tr}(X).$$

Hint: Use that every matrix is similar to an upper triangular matrix, and that both determinant and trace are invariant under change of basis.

4.2.4 Example: Steiner's surface revisited

As we discussed in Section 3.5.4, Steiner's 'Roman surface' is the image of the map

$$\mathbb{R}\mathrm{P}^2 \to \mathbb{R}^3,\ \ (x:y:z) \mapsto \frac{1}{x^2+y^2+z^2}(yz,\ xz,\ xy).$$

(We changed notation from α,β,γ to x,y,z.) Given a point $p \in \mathbb{R}\mathrm{P}^2$, is the map an immersion on an open neighborhood of p? To investigate this question, one can express the map in local charts, and compute the resulting Jacobian matrix. While this approach is perfectly fine, the resulting expressions will become rather complicated. A simpler approach is to consider the composition with the local diffeomorphism $\pi:\ S^2 \to \mathbb{R}\mathrm{P}^2$, given as

$$S^2 \to \mathbb{R}^3,\ \ (x,y,z) \mapsto (yz,\ xz,\ xy).$$

In turn, this map is the restriction $F|_{S^2}$ of the map

$$F:\ \mathbb{R}^3 \to \mathbb{R}^3,\ \ (x,y,z) \mapsto (yz,\ xz,\ xy).$$

We have $T_p(F|_{S^2}) = T_pF|_{T_pS^2}$, hence $\ker(T_p(F|_{S^2})) = \ker(T_pF) \cap T_pS^2$.

Exercise 58.

(a) Compute $T_pF = D_pF$, and find its determinant. Conclude that the kernel is empty except when one of the coordinates is 0.
(b) Suppose p lies in the yz-plane. Find the kernel of D_pF, and $\ker(T_pF) \cap T_pS$. Repeat with p lying in the xz-plane, and the xy-plane.
(c) What are the critical points of the map $\mathbb{R}\mathrm{P}^2 \to \mathbb{R}^3$? (You should have found a total of six critical points.)

4.3 The Tangent Bundle

In the next chapter we define vector fields as a family of tangent vectors, one for each point on the manifold $X_p \in T_pM$, depending smoothly on the base point $p \in M$. This allows us to study dynamics on the manifold, demonstrating the utility of the concept of tangent spaces. Another advantage of this concept is that now we have a vector space associated to each point of the manifold. This allows us to extend many of the natural constructions of linear algebra to manifolds. For example, in Riemannian geometry one defines an inner product $\langle\cdot,\cdot\rangle_p$ on T_pM for each point of the manifold $p \in M$, depending smoothly on the base point $p \in M$. This allows the introduction of many familiar geometric concepts like angles and distances.

The philosophy behind these constructions is that they are defined locally, for each T_pM, but "patch together" globally by requiring smooth dependence on the base point $p \in M$. There are several ways to make the concept of "smooth dependence on base points" rigorous, and we will see some of them in the next chapter. The most elegant is one is simply to use our existing definition of a smooth function between manifolds by turning the collection of all tangent spaces into a manifold. That manifold is called the *tangent bundle*.

Proposition 4.5. *For any manifold M of dimension m, the* tangent bundle

$$TM = \bigsqcup_{p \in M} T_pM$$

(disjoint union of vector spaces) is a manifold of dimension $2m$*. The map*

$$\pi :\ TM \to M$$

taking $v \in T_pM$ to the base point p, is a smooth submersion, with fibers the tangent spaces.

Proof. The idea is simple: Take charts for M, and use the tangent map to get charts for TM. For any open subset U of M, we have

$$TU = \bigsqcup_{p \in U} T_pM = \pi^{-1}(U).$$

(Note $T_pU = T_pM$.) Every chart (U, φ) for M, with $\varphi :\ U \to \mathbb{R}^m$, gives vector space isomorphisms

$$T_p\varphi :\ T_pM \to T_{\varphi(p)}\mathbb{R}^m = \mathbb{R}^m$$

for all $p \in U$. The collection of all maps $T_p\varphi$ for $p \in U$ gives a bijection,

$$T\varphi :\ TU \to \varphi(U) \times \mathbb{R}^m,\ v \mapsto (\varphi(p), (T_p\varphi)(v))$$

for $v \in T_pU \subseteq TU$. The images of these bijections are the open subsets

$$(T\varphi)(TU) = \varphi(U) \times \mathbb{R}^m \subseteq \mathbb{R}^{2m},$$

hence they define charts. We take the collection of all such charts as an atlas for TM:

$$\begin{array}{ccc} TU & \xrightarrow{\ T\varphi\ } & \varphi(U) \times \mathbb{R}^m \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle (u,v)\mapsto u} \\ U & \xrightarrow{\ \varphi\ } & \varphi(U) \end{array}$$

We need to check that the transition maps are smooth. If (V, ψ) is another coordinate chart with $U \cap V \neq \emptyset$, the transition map for $TU \cap TV = T(U \cap V) = \pi^{-1}(U \cap V)$ is given by,

$$T\psi \circ (T\varphi)^{-1} :\ \varphi(U \cap V) \times \mathbb{R}^m \to \psi(U \cap V) \times \mathbb{R}^m. \tag{4.3}$$

But $T_p\psi \circ (T_p\varphi)^{-1} = T_{\varphi(p)}(\psi \circ \varphi^{-1})$ is just the derivative (Jacobian matrix) for the change of coordinates $\psi \circ \varphi^{-1}$; hence (4.3) is given by

$$(x, a) \mapsto \Big((\psi \circ \varphi^{-1})(x),\ D_x(\psi \circ \varphi^{-1})(a)\Big)$$

Since the Jacobian matrix depends smoothly on x, this is a smooth map. This shows that any atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ for M defines an atlas $\{(TU_\alpha, T\varphi_\alpha)\}$ for TM. Taking $\mathscr{A}$ to be countable the atlas for TM is countable. The Hausdorff property is easily checked as well. □

Proposition 4.6. *For any smooth map $F \in C^\infty(M, N)$, the map*

$$TF :\ TM \to TN$$

given on T_pM as the tangent maps $T_pF :\ T_pM \to T_{F(p)}N$, is a smooth map.

Proof. Given $p \in M$, choose charts (U, φ) around p and (V, ψ) around $F(p)$, with $F(U) \subseteq V$. Then $(TU, T\varphi)$ and $(TV, T\psi)$ are charts for TM and TN, respectively, with $TF(TU) \subseteq TV$. Let $\widetilde{F} = \psi \circ F \circ \varphi^{-1} :\ \varphi(U) \to \psi(V)$. The map

$$T\widetilde{F} = T\psi \circ TF \circ (T\varphi)^{-1} :\ \varphi(U) \times \mathbb{R}^m \to \psi(V) \times \mathbb{R}^n$$

is given by

$$(x, a) \mapsto \Big((\widetilde{F})(x),\ D_x(\widetilde{F})(a)\Big).$$

It is smooth, by smooth dependence of the differential $D_x\widetilde{F}$ on the base point. Consequently, TF is smooth, □