

3.4 Submanifolds

Let M be a manifold of dimension m . We will define a k -dimensional submanifold $S \subseteq M$ to be a subset that looks locally like $\mathbb{R}^k \subseteq \mathbb{R}^m$ (which we take to be the coordinate subspace defined by $x^{k+1} = \dots = x^m = 0$).

Definition 3.4. A subset $S \subseteq M$ is called a submanifold of dimension $k \leq m$, if for all $p \in S$ there exists a coordinate chart (U, φ) around p such that

$$\varphi(U \cap S) = \varphi(U) \cap \mathbb{R}^k.$$

Charts (U, φ) of M with this property are called submanifold charts for S .

Remark 3.2.

- (a) A chart (U, φ) such that $U \cap S = \emptyset$ and $\varphi(U) \cap \mathbb{R}^k = \emptyset$ is considered a submanifold chart.
- (b) We stress that the existence of submanifold charts is only required for points p that lie in S . For example, the half-open line $S = (0, \infty)$ is a submanifold of \mathbb{R} (of dimension 1). There does not exist a submanifold chart containing 0, but this is not a problem since $0 \notin S$.

Strictly speaking, a submanifold chart for S is not a chart for S , but is a chart for M which is adapted to S . On the other hand, submanifold charts *restrict* to charts for S , and this may be used to construct an atlas for S :

Proposition 3.4. Suppose S is a submanifold of M . Then S is a k -dimensional manifold in its own right, with atlas consisting of all charts $(U \cap S, \varphi')$ such that (U, φ) is a submanifold chart, and $\varphi' = \pi \circ \varphi|_{U \cap S}$ where $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^k$ is projection onto the first k -coordinates.

Proof. Let (U, φ) and (V, ψ) be two submanifold charts for S . We have to show that the charts $(U \cap S, \varphi')$ and $(V \cap S, \psi')$ are compatible. The map

$$\psi' \circ \varphi'^{-1} : \varphi'(U \cap V \cap S) \rightarrow \psi'(U \cap V \cap S)$$

is smooth, because it is the restriction of $\psi \circ \varphi^{-1} : \varphi(U \cap V \cap S) \rightarrow \psi(U \cap V \cap S)$ to the coordinate subspace \mathbb{R}^k . Likewise its inverse map is smooth. The Hausdorff condition follows because any two distinct points $p, q \in S$, one can take disjoint submanifold charts around p, q . (Just take any submanifold charts, and intersect with the domains of disjoint charts around p, q .)

The proof that S admits a *countable* atlas is a bit technical. We use the following

Exercise 34. Prove that every open subset of \mathbb{R}^m is a union of *rational* ε -balls $B_\varepsilon(x)$, $\varepsilon > 0$. Here, ‘rational’ means that both the center of the ball and its radius are rational: $x \in \mathbb{Q}^m$, $\varepsilon \in \mathbb{Q}$.

Our goal is to construct a countable collection of submanifold charts covering S . (The atlas for S itself is then obtained by restriction.) Start with any countable atlas $(U_\alpha, \varphi_\alpha)$ for M . Given $p \in S \cap U_\alpha$, we can choose a submanifold chart (V, ψ) containing p . Using the above fact, we can choose a rational ε -ball with

$$\varphi(p) \in B_\varepsilon(x) \subseteq \varphi_\alpha(U_\alpha \cap V).$$

This shows that the subsets of the form $\varphi_\alpha^{-1}(B_\varepsilon(x))$, with $B_\varepsilon(x) \subseteq \varphi_\alpha(U_\alpha)$ a rational ε -ball such that $\varphi_\alpha^{-1}(B_\varepsilon(x))$ is contained in *some* submanifold chart, cover all of S . Take these to be the domains of a charts (V_β, ψ_β) , where V_β is one of the $\varphi_\alpha^{-1}(B_\varepsilon(x))$, and ψ_β is the restriction of the coordinate maps of a submanifold chart containing $\varphi_\alpha^{-1}(B_\varepsilon(x))$. You have shown in Exercise 7 that such a set is a chart, and it is easy to see that it is in fact a submanifold chart. Then $\{(V_\beta, \psi_\beta)\}$ is a countable collection of submanifold charts covering S . (Recall that a countable union of countable sets is again countable.) \square

Example 3.8 (Open subsets). The m -dimensional submanifolds of an m -dimensional manifold are exactly the open subsets.

Example 3.9 (Spheres). Let $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\|^2 = 1\}$. Write $x = (x^0, \dots, x^n)$, and regard

$$S^k \subseteq S^n$$

for $k < n$ as the subset where the last $n - k$ coordinates are zero. These are submanifolds: The charts (U_\pm, φ_\pm) for S^n given by stereographic projection

$$\varphi_\pm(x^0, \dots, x^n) = \frac{1}{1 \pm x^0}(x^1, \dots, x^n)$$

are submanifold charts. In fact, the charts U_i^\pm , given by the condition that $\pm x^i > 0$, with φ_i^\pm the projection to the remaining coordinates, are submanifold charts as well.

Example 3.10 (Projective spaces). For $k < n$, regard

$$\mathbb{R}P^k \subseteq \mathbb{R}P^n$$

as the subset of all $(x^0 : \dots : x^n)$ for which $x^{k+1} = \dots = x^n = 0$. These are submanifolds, with the standard charts (U_i, φ_i) for $\mathbb{R}P^n$ as submanifold charts. (Note that the charts U_{k+1}, \dots, U_n don't meet $\mathbb{R}P^k$, but this does not cause a problem.) In fact, the resulting charts for $\mathbb{R}P^k$ obtained by restricting these submanifold charts, are just the standard charts of $\mathbb{R}P^k$. Similarly,

$$\mathbb{C}P^k \subseteq \mathbb{C}P^n$$

are submanifolds, and for $n < n'$ we have $\text{Gr}(k, n) \subseteq \text{Gr}(k, n')$ as a submanifold.

Proposition 3.5. *Let $F : M \rightarrow N$ be a smooth map between manifolds of dimensions m and n . Then*

$$\text{graph}(F) = \{(F(p), p) \mid p \in M\} \subseteq N \times M$$

is a submanifold of $N \times M$, of dimension equal to the dimension of M .

Proof. Given $p \in M$, choose charts (U, φ) around p and (V, ψ) around $F(p)$, with $F(U) \subseteq V$, and let $W = V \times U$. We claim that (W, κ) with

$$\kappa(q, p) = (\varphi(p), \psi(q) - \psi(F(p))) \quad (3.8)$$

is a submanifold chart for $\text{graph}(F) \subseteq N \times M$. Note that this is indeed a chart of $N \times M$, because it is obtained from the product chart $(V \times U, \psi \times \varphi)$ by composition with the diffeomorphism $\psi(V) \times \varphi(U) \rightarrow \varphi(U) \times \psi(V)$, $(v, u) \mapsto (u, v)$, followed by the diffeomorphism

$$\varphi(U) \times \psi(V) \rightarrow \kappa(W), \quad (u, v) \mapsto (u, v - \tilde{F}(u)). \quad (3.9)$$

where $\tilde{F} = \psi \circ F \circ \varphi^{-1}$. Furthermore, the second component in (3.8) vanishes if and only if $F(p) = q$. That is,

$$\kappa(W \cap \text{graph}(F)) = \kappa(W) \cap \mathbb{R}^m$$

as required. \square

Exercise 35. Prove that the map (3.9) is a diffeomorphism by showing that it is smooth and injective, and its Jacobian has determinant one.

This result has the following consequence: If a subset of a manifold, $S \subseteq M$, can be *locally* described as the graph of a smooth map, then S is a submanifold. In more detail, suppose that S can be covered by open sets U , such that for each U there is a diffeomorphism $U \rightarrow P \times Q$ taking $S \cap U$ to the graph of a smooth map $Q \rightarrow P$, then S is a submanifold.

Example 3.11. The 2-torus $S = f^{-1}(0) \subseteq \mathbb{R}^3$, where

$$f(x, y, z) = (\sqrt{x^2 + y^2} - R)^2 + z^2 - r^2$$

is a submanifold of \mathbb{R}^3 , since it can locally be expressed as the graph of a function of x, y , or of y, z , or of x, z .

Exercise 36. 1. Show that on the subset where $z > 0$, S is the graph of the smooth function on the annulus $\{(x, y) \mid (R - r)^2 < x^2 + y^2 < (R + r)^2\}$, given as

$$H(x, y) = \sqrt{r^2 - (\sqrt{x^2 + y^2} - R)^2}.$$

2. Derive a similar formula for $F(y, z)$ on the subset where $x^2 + y^2 < R^2$ and $x > 0$.
3. How many open subsets of this kind (where S is given as the graph of a function of two of the coordinates) are necessary to cover S ?

Example 3.12. More generally, suppose $S \subseteq \mathbb{R}^3$ is given as a level set $S = f^{-1}(0)$ for a smooth map $f \in C^\infty(\mathbb{R}^3)$. (Actually, we only need f to be defined and smooth on an open neighborhood of S .) Let $p \in S$, and suppose

$$\left. \frac{\partial f}{\partial x} \right|_p \neq 0.$$

By the *implicit function theorem* from multivariable calculus, there is an open neighborhood $U \subseteq \mathbb{R}^3$ of p on which the equation $f(x, y, z) = 0$ can be uniquely solved for x . That is,

$$S \cap U = \{(x, y, z) \in U \mid x = F(y, z)\}$$

for a smooth function F , defined on a suitable open subset of \mathbb{R}^2 . This shows that S is a submanifold near p , and in fact we may use y, z as coordinates near p . Similar arguments apply for $\left. \frac{\partial f}{\partial y} \right|_p \neq 0$ or $\left. \frac{\partial f}{\partial z} \right|_p \neq 0$. Hence, if the gradient

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

is non-vanishing at all points $p \in S = f^{-1}(0)$, then S is a 2-dimensional submanifold. Of course, there is nothing special about 2-dimensional submanifolds of \mathbb{R}^3 , and below we will put this discussion in a more general framework.

As we saw, submanifolds S of manifolds M are themselves manifolds. They come with an *inclusion map*

$$i: S \rightarrow M, \quad p \mapsto p,$$

taking any point of S to the same point but viewed as a point of M . Unsurprisingly, we have:

Proposition 3.6. *The inclusion map $i: S \rightarrow M$ is smooth.*

Exercise 37. Prove the proposition.

This shows in particular that if $F \in C^\infty(M, N)$ is a smooth map, then its restriction $F|_S: S \rightarrow N$ is again smooth. Indeed, $F|_S = F \circ i$ is a composition of smooth maps. This is useful in practice, because in such cases there is no need to verify smoothness in the local coordinates of S ! For example, the map $S^2 \rightarrow \mathbb{R}$, $(x, y, z) \mapsto z$ is smooth since it is the restriction of a smooth map $\mathbb{R}^3 \rightarrow \mathbb{R}$ to the submanifold S^2 . A related result, which we leave as an exercise, is the following:

Exercise 38. Let $S \subseteq M$ be a submanifold, with inclusion map i , and let $F: Q \rightarrow S$ be a map from another manifold Q . Then F is smooth if and only if $i \circ F$ is smooth. (In other words, if and only if F is smooth as a map into M .)

For the following proposition, recall that a subset U of a manifold is open if and only if for all $p \in U$, and any coordinate chart (V, ψ) around p , the subset $\psi(U \cap V) \subseteq \mathbb{R}^m$ is open. (This does not depend on the choice of chart.)

Proposition 3.7. *Suppose S is a submanifold of M . Then the open subsets of S for its manifold structure are exactly those of the form $U \cap S$, where U is an open subset of M .*

In other words, the topology of S as a manifold coincides with the ‘subspace topology’ as a subset of the manifold M .

Proof. We have to show:

$$U' \subseteq S \text{ is open} \Leftrightarrow U' = U \cap S \text{ where } U \subseteq M \text{ is open.}$$

“ \Leftarrow ”. Suppose $U \subseteq M$ is open, and let $U' = U \cap S$. For any submanifold chart (V, ψ) , with corresponding chart $(V \cap S, \psi')$ for S (where, as before, $\psi' = \pi \circ \psi|_{V \cap S}$), we have that

$$\psi'((V \cap S) \cap U') = \pi \circ \psi(V \cap S \cap U) = \pi(\psi(U) \cap \psi(V) \cap \mathbb{R}^k).$$

Now, $\psi(U) \cap \psi(V) \cap \mathbb{R}^k$ is the intersection of the open set $\psi(U) \cap \psi(V) \subseteq \mathbb{R}^n$ with the subspace \mathbb{R}^k , hence is open in \mathbb{R}^k . Since submanifold charts cover all of S , this shows that U' is open.

“ \Rightarrow ”. Suppose $U' \subseteq S$ is open in S . Define

$$U = \bigcup_V \psi^{-1}(\psi'(U' \cap V) \times \mathbb{R}^{m-k}) \subseteq M,$$

where the union is over any collection of submanifold charts (V, ψ) that cover all of S . Since U' is open in S , so is $U' \cap V \equiv U' \cap (V \cap S)$, by the previous paragraph. Hence $\psi'(U' \cap V) = \psi'(U' \cap (V \cap S))$ is open in \mathbb{R}^k , and its cartesian product with \mathbb{R}^{m-k} is open in \mathbb{R}^m . The pre-image $\psi^{-1}(\psi(U' \cap V) \times \mathbb{R}^{m-k})$ is thus open in V , hence also in M , and the union over all such sets is open in M . We are now done by the exercise below. \square

Exercise 39. Fill in the last detail of the proof: Check that $U \cap S = U'$.

Remark 3.3. As a consequence, if a manifold M can be realized as a submanifold $M \subseteq \mathbb{R}^n$, then M is compact with respect to its manifold topology if and only if it is compact as a subset of \mathbb{R}^n , if and only if it is a closed and bounded subset of \mathbb{R}^n . This can be used to give quick proofs of the facts that the real or complex projective spaces, as well as the real or complex Grassmannians, are all compact.

Remark 3.4. Sometimes, the result can be used to show that certain subsets are *not* submanifolds. Consider for example the subset

$$S = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\} \subseteq \mathbb{R}^2$$

given as the union of the coordinate axes. If S were a 1-dimensional submanifold, then there would exist an open neighborhood U' of $p = (0, 0)$ in S which is diffeomorphic to an open interval. But for any open subset $U \subseteq \mathbb{R}^2$ containing p , the intersection $U' = U \cap S$ cannot possibly be an open interval, since $(U \cap S) \setminus \{p\}$ has at least four connected components, while removing a point from an open interval gives only two connected components.