2.3 Examples of Manifolds

We will now discuss some basic examples of manifolds. In each case, the manifold structure is given by a finite atlas; hence the countability property is immediate. We will not spend too much time on verifying the Hausdorff property; while it may be done 'by hand', we will later have better ways of doing this.

2.3.1 Spheres

The construction of an atlas for the 2-sphere S^2, by stereographic projection, also works for the n-sphere

$$S^n = \{(x^0,\dots,x^n)|\ (x^0)^2+\cdots+(x^n)^2 = 1\}.$$

Let $U_\pm$ be the subsets obtained by removing $(\mp 1,0,\dots,0)$. Stereographic projection defines bijections $\varphi_\pm :\ U_\pm \to \mathbb{R}^n$, where $\varphi_\pm(x^0, x^1, \dots, x^n) = (u^1,\dots,u^n)$ with

$$u^i = \frac{x^i}{1\pm x^0}.$$

For the transition function one finds (writing $u = (u^1,\dots,u^n)$)

$$(\varphi_- \circ \varphi_+^{-1})(u) = \frac{u}{||u||^2}.$$

We leave it as an exercise to check the details. An equivalent atlas, with $2n+2$ charts, is given by the subsets $U_0^+,\dots,U_n^+,U_0^-,\dots,U_n^-$ where

$$U_j^+ = \{x\in S^n|\ x^j > 0\},\quad U_j^- = \{x\in S^n|\ x^j < 0\}$$

for $j = 0,\dots,n$, with $\varphi_j^\pm :\ U_j^\pm \to \mathbb{R}^n$ the projection to the j-th coordinate plane (in other words, omitting the j-th component x^j):

$$\varphi_j^\pm(x^0,\dots,x^n) = (x^0,\dots,x^{j-1},x^{j+1},\dots,x^n).$$

2.3.2 Real projective spaces

The n-dimensional projective space $\mathbb{R}\mathrm{P}^n$, is the set of all lines $\ell \subseteq \mathbb{R}^{n+1}$. It may also be regarded as a quotient space[‡]

$$\mathbb{R}\mathrm{P}^n = (\mathbb{R}^{n+1}\backslash\{0\})/\sim$$

for the equivalence relation

$$x\sim x' \Leftrightarrow \exists\lambda \in \mathbb{R}\backslash\{0\} :\ x' = \lambda x.$$

[‡] See the appendix to this chapter for some background on quotient spaces.

Indeed, any $x \in \mathbb{R}^{n+1}\backslash\{0\}$ determines a line, while two points x, x' determine the same line if and only if they agree up to a non-zero scalar multiple. The equivalence class of $x = (x^0, \ldots, x^n)$ under this relation is commonly denoted

$$[x] = (x^0 : \ldots : x^n).$$

$\mathbb{R}\mathrm{P}^n$ has a *standard atlas*

$$\mathscr{A} = \{(U_0, \varphi_0), \ldots, (U_n, \varphi_n)\}$$

defined as follows. For $j = 0, \ldots, n$, let

$$U_j = \{(x^0 : \ldots : x^n) \in \mathbb{R}\mathrm{P}^n |\; x^j \neq 0\}$$

be the set for which the j-th coordinate is non-zero, and put

$$\varphi_j : \; U_j \to \mathbb{R}^n, \;\; (x^0 : \ldots : x^n) \mapsto \Big(\frac{x^0}{x^j}, \ldots, \frac{x^{j-1}}{x^j}, \frac{x^{j+1}}{x^j}, \ldots, \frac{x^n}{x^j}\Big).$$

This is well-defined, since the quotients do not change when all x^i are multiplied by a fixed scalar. Put differently, given an element $[x] \in \mathbb{R}\mathrm{P}^n$ for which the j-th component x^j is non-zero, we first rescale the representative x to make the j-th component equal to 1, and then use the remaining components as our coordinates. As an example (with $n = 2$),

$$\varphi_1(7 : 3 : 2) = \varphi_1\Big(\frac{7}{3} : 1 : \frac{2}{3}\Big) = \Big(\frac{7}{3}, \frac{2}{3}\Big).$$

From this description, it is immediate that φ_j is a bijection from U_j onto $\mathbb{R}^n$, with inverse map

$$\varphi_j^{-1}(u^1, \ldots, u^n) = (u^1 : \ldots : u^j : 1 : u^{j+1} : \ldots : u^n).$$

Geometrically, viewing $\mathbb{R}\mathrm{P}^n$ as the set of lines in $\mathbb{R}^{n+1}$, the subset $U_j \subseteq \mathbb{R}\mathrm{P}^n$ consists of those lines ℓ which intersect the affine hyperplane

$$H_j = \{x \in \mathbb{R}^{n+1} |\; x^j = 1\},$$

and the map φ_j takes such a line ℓ to its unique point of intersection $\ell \cap H_j$, followed by the identification $H_j \cong \mathbb{R}^n$ (dropping the coordinate $x^j = 1$).

Let us verify that $\mathscr{A}$ is indeed an atlas. Clearly, the domains U_j cover $\mathbb{R}\mathrm{P}^n$, since any element $[x] \in \mathbb{R}\mathrm{P}^n$ has at least one of its components non-zero. For $i \neq j$, the intersection $U_i \cap U_j$ consists of elements x with the property that both components x^i, x^j are non-zero.

Exercise 8. Compute the transition maps $\varphi_i \circ \varphi_j^{-1}$, and verify they are smooth. (Hint: You will need to distinguish between the cases $i < j$ and $i > j$.)

To complete the proof that this atlas (or the unique maximal atlas containing it) defines a manifold structure, it remains to check the Hausdorff property.

This can be done with the help of Lemma 2.2, but we postpone the proof since we will soon have a simple argument in terms of smooth functions. See Proposition 3.1 below.

Remark 2.4. In low dimensions, we have that $\mathbb{R}\mathrm{P}^0$ is just a point, while $\mathbb{R}\mathrm{P}^1$ is a circle.

Remark 2.5. Geometrically, U_i consists of all lines in $\mathbb{R}^{n+1}$ meeting the affine hyperplane H_i, hence its complement consists of all lines that are parallel to H_i, i.e., the lines in the coordinate subspace defined by $x^i = 0$. The set of such lines is $\mathbb{R}\mathrm{P}^{n-1}$. In other words, the complement of U_i in $\mathbb{R}\mathrm{P}^n$ is identified with $\mathbb{R}\mathrm{P}^{n-1}$.

Thus, as sets, $\mathbb{R}\mathrm{P}^n$ is a disjoint union

$$\mathbb{R}\mathrm{P}^n = \mathbb{R}^n \sqcup \mathbb{R}\mathrm{P}^{n-1},$$

where $\mathbb{R}^n$ is identified (by the coordinate map φ_i) with the open subset U_n, and $\mathbb{R}\mathrm{P}^{n-1}$ with its complement. Inductively, we obtain a decomposition

$$\mathbb{R}\mathrm{P}^n = \mathbb{R}^n \sqcup \mathbb{R}^{n-1} \sqcup \cdots \sqcup \mathbb{R} \sqcup \mathbb{R}^0,$$

where $\mathbb{R}^0 = \{0\}$. At this stage, it is simply a decomposition into subsets; later it will be recognized as a decomposition into submanifolds.

Exercise 9. Find an identification of the space of rotations in $\mathbb{R}^3$ with the 3-dimensional projective space $\mathbb{R}\mathrm{P}^3$.
(Hint: Associate to any $v \in \mathbb{R}^3$ a rotation, as follows: If $v = 0$, take the trivial rotation; if $v \neq 0$, take the rotation by an angle $||v||$ around the oriented axis determined by v. Note that for $||v|| = \pi$, the vectors v and $-v$ determine the same rotation.)

2.3.3 Complex projective spaces

In a similar fashion to the real projective space, one can define a *complex projective space* $\mathbb{C}\mathrm{P}^n$ as the set of complex 1-dimensional subspaces of $\mathbb{C}^{n+1}$. If we identify $\mathbb{C}$ with $\mathbb{R}^2$, thus $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$, we have

$$\mathbb{C}\mathrm{P}^n = (\mathbb{C}^{n+1}\backslash\{0\})/\sim$$

where the equivalence relation is $z \sim z'$ if and only if there exists a complex λ with $z' = \lambda z$. (Note that the scalar λ is then unique, and is non-zero.) Alternatively, letting $S^{2n+1} \subseteq \mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ be the 'unit sphere' consisting of complex vectors of length $||z|| = 1$, we have

$$\mathbb{C}\mathrm{P}^n = S^{2n+1}/\sim,$$

where $z' \sim z$ if and only if there exists a complex number λ with $z' = \lambda z$. (Note that the scalar λ is then unique, and has absolute value 1.) One defines charts (U_j, φ_j) similarly to those for the real projective space:

$$U_j = \left\{(z^0 : \ldots : z^n) \,|\, z^j \neq 0\right\}, \quad \varphi_j :\ U_j \to \mathbb{C}^n = \mathbb{R}^{2n},$$

$$\varphi_j(z^0 : \ldots : z^n) = \Big(\frac{z^0}{z^j}, \ldots, \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j}, \ldots, \frac{z^n}{z^j}\Big).$$

The transition maps between charts are given by similar formulas as for $\mathbb{R}\mathrm{P}^n$ (just replace x with z); they are smooth maps between open subsets of $\mathbb{C}^n = \mathbb{R}^{2n}$. Thus $\mathbb{C}\mathrm{P}^n$ is a smooth manifold of dimension $2n$§. As with $\mathbb{R}\mathrm{P}^n$ there is a decomposition

$$\mathbb{C}\mathrm{P}^n = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C} \sqcup \mathbb{C}^0.$$

2.3.4 Grassmannians

The set $\mathrm{Gr}(k,n)$ of all k-dimensional subspaces of $\mathbb{R}^n$ is called the *Grassmannian of k-planes in* $\mathbb{R}^n$. (Named after *Hermann Grassmann* (1809-1877).)

As a special case, $\mathrm{Gr}(1,n) = \mathbb{R}\mathrm{P}^{n-1}$.

We will show that for general k, the Grassmannian is a manifold of dimension

$$\dim(\mathrm{Gr}(k,n)) = k(n-k).$$

An atlas for $\mathrm{Gr}(k,n)$ may be constructed as follows. The idea is to present linear subspaces of dimension k as graphs of linear maps from $\mathbb{R}^k$ to $\mathbb{R}^{n-k}$. Here $\mathbb{R}^k$ is viewed as the coordinate subspace corresponding to a choice of k components from $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$, and $\mathbb{R}^{n-k}$ the coordinate subspace for the remaining coordinates. To make it precise, we introduce some notation.

For any subset $I \subseteq \{1, \ldots, n\}$ of the set of indices, let

$$I' = \{1, \ldots, n\} \backslash I$$

be its complement. Let $\mathbb{R}^I \subseteq \mathbb{R}^n$ be the coordinate subspace

$$\mathbb{R}^I = \{x \in \mathbb{R}^n |\; x^i = 0 \text{ for all } i \in I'\}.$$

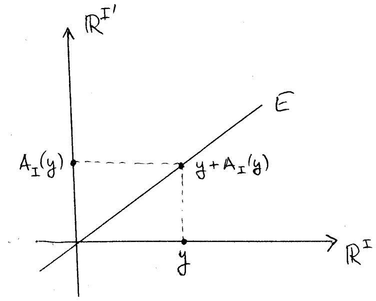
If I has cardinality $|I| = k$, then $\mathbb{R}^I \in \mathrm{Gr}(k,n)$. Note that $\mathbb{R}^{I'} = (\mathbb{R}^I)^\perp$. Let

$$U_I = \{E \in \mathrm{Gr}(k,n) |\, E \cap \mathbb{R}^{I'} = \{0\}\}.$$

§ The transition maps are not only smooth but even holomorphic, making $\mathbb{C}\mathrm{P}^n$ into an example of a *complex manifold* (of complex dimension n).

Each $E \in U_I$ is described as the graph of a unique linear map $A_I : \mathbb{R}^I \to \mathbb{R}^{I'}$, that is,

$$E = \{y + A_I(y) \,|\, y \in \mathbb{R}^I\}.$$



Exercise 10. Let E be as above. Show that there is a unique linear map $A_I : \mathbb{R}^I \to \mathbb{R}^{I'}$ such that $E = \{y + A_I(y) \,|\, y \in \mathbb{R}^I\}$.

This gives a bijection

$$\varphi_I : U_I \to \mathrm{Hom}(\mathbb{R}^I, \mathbb{R}^{I'}),\ E \mapsto \varphi_I(E) = A_I,$$

where $\mathrm{Hom}(F, F')$ denotes the space of linear maps from a vector space F to a vector space F'. Note $\mathrm{Hom}(\mathbb{R}^I, \mathbb{R}^{I'}) \cong \mathbb{R}^{k(n-k)}$, because the bases of $\mathbb{R}^I$ and $\mathbb{R}^{I'}$ identify the space of linear maps with $(n-k) \times k$-matrices, which in turn is just $\mathbb{R}^{k(n-k)}$ by listing the matrix entries. In terms of A_I, the subspace $E \in U_I$ is the range of the injective linear map

$$\begin{pmatrix} 1 \\ A_I \end{pmatrix} :\ \mathbb{R}^I \to \mathbb{R}^I \oplus \mathbb{R}^{I'} \cong \mathbb{R}^n \tag{2.1}$$

where we write elements of $\mathbb{R}^n$ as column vectors.

To check that the charts are compatible, suppose $E \in U_I \cap U_J$, and let A_I and A_J be the linear maps describing E in the two charts. We have to show that the map

$$\varphi_J \circ \varphi_I^{-1} :\ \mathrm{Hom}(\mathbb{R}^I, \mathbb{R}^{I'}) \to \mathrm{Hom}(\mathbb{R}^J, \mathbb{R}^{J'}),\ \ A_I = \varphi_I(E) \mapsto A_J = \varphi_J(E)$$

is smooth. By assumption, E is described as the range of (2.1) and also as the range of a similar map for J. Here we are using the identifications $\mathbb{R}^I \oplus \mathbb{R}^{I'} \cong \mathbb{R}^n$ and $\mathbb{R}^J \oplus \mathbb{R}^{J'} \cong \mathbb{R}^n$. It is convenient to describe everything in terms of $\mathbb{R}^J \oplus \mathbb{R}^{J'}$. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} :\ \mathbb{R}^I \oplus \mathbb{R}^{I'} \to \mathbb{R}^J \oplus \mathbb{R}^{J'}$$

be the matrix corresponding to the identification $\mathbb{R}^I \oplus \mathbb{R}^{I'} \to \mathbb{R}^n$ followed by the inverse of $\mathbb{R}^J \oplus \mathbb{R}^{J'} \to \mathbb{R}^n$. For example, c is the inclusion $\mathbb{R}^I \to \mathbb{R}^n$ as the corresponding coordinate subspace, followed by projection to the coordinate subspace $\mathbb{R}^{J'}$. [¶] We then get the condition that the injective linear maps

[¶] Put differently, the matrix is the permutation matrix 'renumbering' the coordinates of $\mathbb{R}^n$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ A_I \end{pmatrix} : \mathbb{R}^I \to \mathbb{R}^J \oplus \mathbb{R}^{J'}, \quad \begin{pmatrix} 1 \\ A_J \end{pmatrix} : \mathbb{R}^J \to \mathbb{R}^J \oplus \mathbb{R}^{J'}$$

have the same range. In other words, there is an isomorphism $S: \mathbb{R}^I \to \mathbb{R}^J$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ A_I \end{pmatrix} = \begin{pmatrix} 1 \\ A_J \end{pmatrix} S$$

as maps $\mathbb{R}^I \to \mathbb{R}^J \oplus \mathbb{R}^{J'}$. We obtain

$$\begin{pmatrix} a + bA_I \\ c + dA_I \end{pmatrix} = \begin{pmatrix} S \\ A_J S \end{pmatrix}$$

Using the first row of this equation to eliminate the second row of this equation, we obtain the formula

$$A_J = (c + dA_I)(a + bA_I)^{-1}.$$

The dependence of the right hand side on the matrix entries of A_I is smooth, by Cramer's formula for the inverse matrix. It follows that the collection of all $\varphi_I : U_I \to \mathbb{R}^{k(n-k)}$ defines on $\mathrm{Gr}(k,n)$ the structure of a manifold of dimension $k(n-k)$. The number of charts of this atlas equals the number of subsets $I \subseteq \{1,\ldots,n\}$ of cardinality k, that is, it is equal to $\binom{n}{k}$. (The Hausdorff property may be checked in a similar fashion to $\mathbb{R}\mathrm{P}^n$. Alternatively, given distinct $E_1, E_2 \in \mathrm{Gr}(k,n)$, choose a subspace $F \in \mathrm{Gr}(k,n)$ such that $F^\perp$ has zero intersection with both E_1, E_2. (Such a subspace always exists.) One can then define a chart (U, φ), where U is the set of subspaces E transverse to $F^\perp$, and φ realizes any such map as the graph of a linear map $F \to F^\perp$. Thus $\varphi: U \to \mathrm{Hom}(F, F^\perp)$. As above, we can check that this is compatible with all the charts (U_I, φ_I). Since both E_1, E_2 are in this chart U, we are done by Lemma 2.2.)

Exercise 11. Prove the parenthetical remark above: If $E_1, E_2 \in \mathrm{Gr}(k,n)$ are distinct, show that there exists $F \in \mathrm{Gr}(k,n)$ such that $F^\perp \cap E_1 = F^\perp \cap E_2 = \{0\}$.

Remark 2.6. As already mentioned, $\mathrm{Gr}(1,n) = \mathbb{R}\mathrm{P}^{n-1}$. One can check that our system of charts in this case is the standard atlas for $\mathbb{R}\mathrm{P}^{n-1}$.

Exercise 12. This is a preparation for the following remark. Recall that a linear map $\Pi: \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal projection onto some subspace $E \subseteq \mathbb{R}^n$ if $\Pi(x) = x$ for $x \in E$ and $\Pi(x) = 0$ for $x \in E^\perp$. Show that a square matrix $P \in \mathrm{Mat}_{\mathbb{R}}(n)$ is the matrix of an orthogonal projection if and only if it has the properties

$$P^\top = P \text{ and } PP = P.$$

What is the matrix of the orthogonal projection onto $E^\perp$?

Remark 2.7. For any k-dimensional subspace $E \subseteq \mathbb{R}^n$, let $\Pi^E : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map given by orthogonal projection onto E, and let $P_E \in \mathrm{Mat}_{\mathbb{R}}(n)$ be its matrix. By the exercise,

$$P_E^\top = P_E, \;\; P_E P_E = P_E,$$

Conversely, any square matrix P with the properties $P^\top = P$, $PP = P$ with $\mathrm{rank}(P) = k$ is the orthogonal projection onto a subspace $\{Px |\, x \in \mathbb{R}^n\} \subseteq \mathbb{R}^n$. This identifies the Grassmannian $\mathrm{Gr}(k,n)$ with the set of orthogonal projections of rank k. In summary, we have an inclusion

$$\mathrm{Gr}(k,n) \hookrightarrow \mathrm{Mat}_{\mathbb{R}}(n) \cong \mathbb{R}^{n^2}, \quad E \mapsto P_E.$$

By construction, this inclusion take values in the subspace $\mathrm{Sym}_{\mathbb{R}}(n) \cong \mathbb{R}^{n(n+1)/2}$ of *symmetric* $n \times n$-matrices.

Remark 2.8. For all k, there is an identification $\mathrm{Gr}(k,n) \cong \mathrm{Gr}(n-k,n)$ (taking a k-dimensional subspace to the orthogonal subspace).

Remark 2.9. Similar to $\mathbb{R}\mathrm{P}^2 = S^2/\sim$, the quotient modulo antipodal identification, one can also consider

$$M = (S^2 \times S^2)/\sim$$

the quotient space by the equivalence relation

$$(x,x') \sim (-x,-x').$$

It turns out that this manifold M is the same as $\mathrm{Gr}(2,4)$, where ‘the same’ is meant in the sense that there is a bijection of sets identifying the atlases. Note that this is the first genuinely new manifold, since $\mathrm{Gr}(1,4) = \mathbb{R}\mathrm{P}^3$, and $\mathrm{Gr}(3,4) \cong \mathrm{Gr}(1,4)$ by the previous remark.
Question: What about the other $\mathrm{Gr}(k,n)$ with $n \leq 4$?

2.3.5 Complex Grassmannians

Similar to the case of projective spaces, one can also consider the *complex Grassmannian* $\mathrm{Gr}_{\mathbb{C}}(k,n)$ of complex k-dimensional subspaces of $\mathbb{C}^n$. It is a manifold of dimension $2k(n-k)$, which can also be regarded as a complex manifold of complex dimension $k(n-k)$.