Partitions of unity allow us to go from local to global, i.e. to build a global object on a manifold by building it on each open set of a cover, smoothly tapering each local piece so it is compactly supported in each open set, and then taking a sum over open sets. This is a very flexible operation which uses the properties of smooth functions—it will not work for complex manifolds, for example. Our main example of such a passage from local to global is to build a global map from a manifold to $\mathbb{R}^N$ which is an embedding, a result first proved by Whitney.

**Definition 4.1.** A collection of subsets $\{U_\alpha\}$ of the topological space $M$ is called **locally finite** when each point $x \in M$ has a neighbourhood $V$ intersecting only finitely many of the $U_\alpha$.

**Definition 4.2.** A covering $\{V_\alpha\}$ is a **refinement** of the covering $\{U_\beta\}$ when each $V_\alpha$ is contained in some $U_\beta$.

**Lemma 4.3.** Any open covering $\{A_\alpha\}$ of a topological manifold has a countable, locally finite refinement $\{(U_i, \varphi_i)\}$ by coordinate charts such that $\varphi_i(U_i) = B(0,3)$ and $\{V_i = \varphi_i^{-1}(B(0,1))\}$ is still a covering of $M$. We will call such a cover a regular covering. In particular, any topological manifold is paracompact (i.e. every open cover has a locally finite refinement).

**Proof.** If $M$ is compact, the proof is easy: choosing coordinates around any point $x \in M$, we can translate and rescale to find a covering of $M$ by a refinement of the type desired, and choose a finite subcover, which is obviously locally finite.

For a general manifold, we note that by second countability of $M$, there is a countable basis of coordinate neighbourhoods and each of these charts is a countable union of open sets $P_i$ with $P_i$ compact. Hence $M$ has a countable basis $\{P_i\}$ such that $P_i$ is compact.

Using these, we may define an increasing sequence of compact sets which exhausts $M$: let $K_1 = P_1$, and

$$K_{i+1} = P_1 \cup \cdots \cup P_r,$$

where $r > 1$ is the first integer with $K_1 \subset P_1 \cup \cdots \cup P_r$.

Now note that $M$ is the union of ring-shaped sets $K_i \setminus K_{i-1}$, each of which is compact. If $p \in A_\alpha$, then $p \in K_{i+1} \setminus K_i$ for some $i$. Now choose a coordinate neighbourhood $(U_{p,\alpha}, \varphi_{p,\alpha})$ with $U_{p,\alpha} \subset K_{i+2} \setminus K_{i-1}$ and $\varphi_{p,\alpha}(U_{p,\alpha}) = B(0,3)$ and define $V_{p,\alpha} = \varphi^{-1}(B(0,1))$.

Letting $p, \alpha$ vary, these neighbourhoods cover the compact set $K_{i+1} \setminus K_i$ without leaving the band $K_{i+2} \setminus K_{i-1}$. Choose a finite subcover $V_{i,k}$ for each $i$. Then $(U_{i,k}, \varphi_{i,k})$ is the desired locally finite refinement.

**Definition 4.4.** A smooth partition of unity is a collection of smooth non-negative functions $\{f_\alpha : M \to \mathbb{R}\}$ such that

i) $\{\text{supp} f_\alpha = f_\alpha^{-1}(\mathbb{R}\setminus\{0\})\}$ is locally finite,

ii) $\sum_\alpha f_\alpha(x) = 1 \quad \forall x \in M$, hence the name.
A partition of unity is **subordinate** to an open cover \( \{ U_i \} \) when \( \forall x, \text{supp} f_i \subset U_i \) for some \( i \).

**Theorem 4.5.** Given a regular covering \( \{(U_i, \varphi_i)\} \) of a manifold, there exists a partition of unity \( \{ f_i \} \) subordinate to it with \( f_i > 0 \) on \( V_i \) and \( \text{supp} f_i \subset \varphi_i^{-1}(B(0, 2)) \).

**Proof.** A bump function is a smooth non-negative real-valued function \( \tilde{g} \) on \( \mathbb{R}^n \) with \( \tilde{g}(x) = 1 \) for \( ||x|| \leq 1 \) and \( \tilde{g}(x) = 0 \) for \( ||x|| \geq 2 \). For instance, take
\[
\tilde{g}(x) = \frac{h(2 - ||x||)}{h(2 - ||x||) + h(||x|| - 1)},
\]
for \( h(t) \) given by \( e^{-1/t} \) for \( t > 0 \) and 0 for \( t < 0 \).

Having this bump function, we can produce non-negative bump functions on the manifold \( g_i = \tilde{g} \circ \varphi_i \), which have support \( \text{supp} g_i \subset \varphi_i^{-1}(B(0, 2)) \) and take the value +1 on \( V_i \). Finally we define our partition of unity via
\[
f_i = \frac{g_i}{\sum_j g_j}, \quad i = 1, 2, \ldots
\]
\( \square \)

### 4.1 Whitney embedding

We now investigate the embedding of arbitrary smooth manifolds as regular submanifolds of \( \mathbb{R}^k \).

**Theorem 4.6** (Compact Whitney embedding in \( \mathbb{R}^N \)). Any compact manifold may be embedded in \( \mathbb{R}^N \) for sufficiently large \( N \).

**Proof.** Let \( \{(U_i \supset V_i, \varphi_i)\}_{i=1}^k \) be a finite regular covering, which exists by compactness. Choose a partition of unity \( \{f_1, \ldots, f_k\} \) as in Theorem 4.5 and define the following “zoom-in” maps \( M \rightarrow \mathbb{R}^{\dim M} \):
\[
\tilde{\varphi}_i(x) = \begin{cases} f_i(x) \varphi_i(x) & x \in U_i, \\ 0 & x \notin U_i. \end{cases}
\]
Then define a map \( \Phi : M \rightarrow \mathbb{R}^{k(\dim M + 1)} \) which zooms simultaneously into all neighbourhoods, with extra information to guarantee injectivity:
\[
\Phi(x) = (\tilde{\varphi}_1(x), \ldots, \tilde{\varphi}_k(x), f_1(x), \ldots, f_k(x)).
\]
Note that \( \Phi(x) = \Phi(x') \) implies that for some \( i, f_i(x) = f_i(x') \neq 0 \) and hence \( x, x' \in U_i \). This then implies that \( \varphi_i(x) = \varphi_i(x') \), implying \( x = x' \). Hence \( \Phi \) is injective.

We now check that \( D\Phi \) is injective, which will show that it is an injective immersion. At any point \( x \) the differential sends \( v \in T_x M \) to the following vector in \( \mathbb{R}^{\dim M} \times \cdots \times \mathbb{R}^{\dim M} \times \mathbb{R} \times \cdots \times \mathbb{R} \):
\[
(Df_1(v)\varphi_1(x) + f_1(x)D\varphi_1(v), \ldots, Df_k(v)\varphi_k(x) + f_k(x)D\varphi_k(v), Df_1(v), \ldots, Df_k(v))
\]
But this vector cannot be zero. Hence we see that \( \Phi \) is an immersion.

But an injective immersion from a compact space must be an embedding: view \( \Phi \) as a bijection onto its image. We must show that \( \Phi^{-1} \) is
continuous, i.e. that \( \Phi \) takes closed sets to closed sets. If \( K \subseteq M \) is closed, it is also compact and hence \( \Phi(K) \) must be compact, hence closed (since the target is Hausdorff).

**Theorem 4.7** (Compact Whitney embedding in \( \mathbb{R}^{2n+1} \)). Any compact \( n \)-manifold may be embedded in \( \mathbb{R}^{2n+1} \).

**Proof.** Begin with an embedding \( \Phi : M \rightarrow \mathbb{R}^N \) and assume \( N > 2n + 1 \). We then show that by projecting onto a hyperplane it is possible to obtain an embedding to \( \mathbb{R}^{N-1} \).

A vector \( v \in S^{N-1} \subseteq \mathbb{R}^N \) defines a hyperplane (the orthogonal complement) and let \( P_v : \mathbb{R}^N \rightarrow \mathbb{R}^{N-1} \) be the orthogonal projection to this hyperplane. We show that the set of \( v \) for which \( \Phi_v = P_v \circ \Phi \) fails to be an embedding is a set of measure zero, hence that it is possible to choose \( v \) for which \( \Phi_v \) is an embedding.

\( \Phi_v \) fails to be an embedding exactly when \( \Phi_v \) is not injective or \( D\Phi_v \) is not injective at some point. Let us consider the two failures separately:

If \( v \) is in the image of the map \( \beta_1 : (M \times M) \setminus \Delta_M \rightarrow S^{N-1} \) given by

\[
\beta_1(p_1, p_2) = \frac{\Phi(p_2) - \Phi(p_1)}{||\Phi(p_2) - \Phi(p_1)||},
\]

then \( \Phi_v \) will fail to be injective. Note however that \( \beta_1 \) maps a \( 2n \)-dimensional manifold to a \( N-1 \)-manifold, and if \( N > 2n + 1 \) then baby Sard’s theorem implies the image has measure zero.

The immersion condition is a local one, which we may analyze in a chart \((U, \varphi)\). \( \Phi_v \) will fail to be an immersion in \( U \) precisely when \( v \) coincides with a vector in the normalized image of \( D(\Phi \circ \varphi^{-1}) \) where

\[
\Phi \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N.
\]

Hence we have a map (letting \( N(w) = ||w|| \))

\[
\frac{D(\Phi \circ \varphi^{-1})}{N \circ D(\Phi \circ \varphi^{-1})} : U \times S^{n-1} \rightarrow S^{N-1}.
\]

The image has measure zero as long as \( 2n - 1 < N - 1 \), which is certainly true since \( 2n < N - 1 \). Taking union over countably many charts, we see that immersion fails on a set of measure zero in \( S^{N-1} \).

Hence we see that \( \Phi_v \) fails to be an embedding for a set of \( v \in S^{N-1} \) of measure zero. Hence we may reduce \( N \) all the way to \( N = 2n + 1 \).

**Corollary 4.8.** We see from the proof that if we do not require injectivity but only that the manifold be immersed in \( \mathbb{R}^N \), then we can take \( N = 2n \) instead of \( 2n + 1 \).

We now use Whitney embedding to prove the existence of tubular neighbourhoods for submanifolds of \( \mathbb{R}^N \), a key point in proving genericity of transversality. Tubular neighbourhoods also exist for submanifolds of any manifold, but we leave this corollary for the reader.

If \( Y \subseteq \mathbb{R}^N \) is an embedded submanifold, the normal space at \( y \in Y \) is defined by \( N_y Y = \{ v \in \mathbb{R}^N : \langle v, T_y Y \rangle = 0 \} \). The collection of all normal spaces of all points in \( Y \) is called the normal bundle:

\[
NY = \{(y, v) \in Y \times \mathbb{R}^N : v \in N_y Y \}.
\]
**Proposition 4.9.** \( NY \subset \mathbb{R}^N \times \mathbb{R}^N \) is an embedded submanifold of dimension \( N \).

**Proof.** Given \( y \in Y \), choose coordinates \((u^1, \ldots, u^N)\) in a neighbourhood \( U \subset \mathbb{R}^N \) of \( y \) so that \( Y \cap U = \{u^{N+1} = \cdots = u^N = 0\} \). Define \( \Phi : U \times \mathbb{R}^N \rightarrow \mathbb{R}^{N-n} \times \mathbb{R}^n \) via

\[
\Phi(x, v) = (u^{N+1}(x), \ldots, u^N(x), (v, \frac{\partial}{\partial u^1}(x)), \ldots, (v, \frac{\partial}{\partial u^n}(x))),
\]

so that \( \Phi^{-1}(0) \) is precisely \( NY \cap (U \times \mathbb{R}^N) \). We then show that 0 is a regular value: observe that, writing \( v \) in terms of its components \( v^j \frac{\partial}{\partial x^j} \)

in the standard basis for \( \mathbb{R}^N \),

\[
\langle v, \frac{\partial}{\partial v^i} \rangle = \langle v^j \frac{\partial}{\partial x^j}(u(x)) \frac{\partial}{\partial x^i}(u(x)) \rangle = \sum_{j=1}^N v^j \frac{\partial u^j}{\partial x^i}(u(x))
\]

Therefore the Jacobian of \( \Phi \) is the \(((N-n) + n) \times (N + N)\) matrix

\[
D\Phi(x) = \begin{pmatrix}
\frac{\partial u^i}{\partial x^j}(x) & 0 \\
* & \frac{\partial u^i}{\partial x^j}(u(x))
\end{pmatrix}
\]

The \( N \) rows of this matrix are linearly independent, proving \( \Phi \) is a submersion. \( \square \)

The normal bundle \( NY \) contains \( Y \cong Y \times \{0\} \) as a regular submanifold, and is equipped with a smooth map \( \pi : NY \rightarrow Y \) sending \((y, v) \mapsto y \). The map \( \pi \) is a surjective submersion and is the bundle projection. The vector spaces \( \pi^{-1}(y) \) for \( y \in Y \) are called the fibers of the bundle and \( NY \) is an example of a vector bundle.

We may take advantage of the embedding in \( \mathbb{R}^N \) to define a smooth map \( E : NY \rightarrow \mathbb{R}^N \) via

\[
E(x, v) = x + v.
\]

**Definition 4.10.** A tubular neighbourhood of the embedded submanifold \( Y \subset \mathbb{R}^N \) is a neighbourhood \( U \) of \( Y \) in \( \mathbb{R}^N \) that is the diffeomorphic image under \( E \) of an open subset \( V \subset NY \) of the form

\[
V = \{(y, v) \in NY : |v| < \delta(y)\},
\]

for some positive continuous function \( \delta : M \rightarrow \mathbb{R} \).

If \( U \subset \mathbb{R}^N \) is such a tubular neighbourhood of \( Y \), then there does exist a positive continuous function \( \epsilon : Y \rightarrow \mathbb{R} \) such that \( U_\epsilon = \{x \in \mathbb{R}^N : \exists y \in Y \text{ with } |x - y| < \epsilon(y)\} \) is contained in \( U \). This is simply

\[
\epsilon(y) = \sup\{r \mid B(y, r) \subset U\},
\]

which is continuous since \( \forall \epsilon > 0, \exists x \in U \) for which \( \epsilon(y) \leq |x - y| + \epsilon \). For any other \( y' \in Y \), this is \( \leq |y - y'| + |x - y'| + \epsilon \). Since \( |x - y'| \leq \epsilon(y') \), we have \( |\epsilon(y) - \epsilon(y')| \leq |y - y'| + \epsilon \).

**Theorem 4.11** (Tubular neighbourhood theorem). Every regular submanifold of \( \mathbb{R}^N \) has a tubular neighbourhood. 


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Proof. First we show that $E$ is a local diffeomorphism near $y \in Y \subset NY$.
If $\iota$ is the embedding of $Y$ in $\mathbb{R}^N$, and $\iota' : Y \to NY$ is the embedding in
the normal bundle, then $E \circ \iota' = \iota$, hence we have $DE \circ D\iota' = D\iota$, showing
that the image of $DE(y)$ contains $T_yY$. Now if $\iota$ is the embedding of $N_yY$
in $\mathbb{R}^N$, and $\iota' : N_yY \to NY$ is the embedding in the normal bundle,
then $E \circ \iota' = \iota$. Hence we see that the image of $DE(y)$ contains $N_yY$, and
hence the image is all of $T_y\mathbb{R}^N$. Hence $E$ is a diffeomorphism on some
neighbourhood

$$V_\delta(y) = \{(y',v') \in NY : |y' - y| < \delta, |v'| < \delta\}, \quad \delta > 0.$$ 

Now for $y \in Y$ let $r(y) = \sup \{\delta : E|_{V_\delta(y)} \text{ is a diffeomorphism} \}$ if this is
$\leq 1$ and let $r(y) = 1$ otherwise. The function $r(y)$ is continuous, since if
$|y - y'| < r(y)$, then $V_\delta(y') \subset V_{r(y)}(y)$ for $\delta = r(y) - |y - y'|$. This means
that $r(y') \geq \delta$, i.e. $r(y) - r(y') \leq |y - y'|$. Switching $y$ and $y'$, this remains
true, hence $|r(y) - r(y')| \leq |y - y'|$, yielding continuity.

Finally, let $V = \{(y,v) \in NY : |v| < \frac{1}{2} r(y)\}$. We show that $E$
is injective on $V$. Suppose $(y,v), (y',v') \in V$ are such that $E(y,v) = E(y',v')$, and suppose wlog $r(y') \leq r(y)$. Then since $y + v = y' + v'$, we have

$$|y - y'| = |v - v'| \leq |v| + |v'| \leq \frac{1}{2} r(y) + \frac{1}{2} r(y') \leq r(y).$$

Hence $y, y'$ are in $V_{r(y)}(y)$, on which $E$ is a diffeomorphism. The required
neighbourhood is then $U = E(V)$. \qed