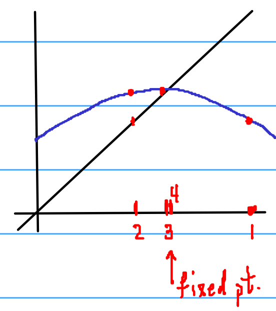
Inverse function theorem

If $f:(M,p)\to(N,q)$ is a smooth map of n-manifolds such that $Df(p): T_pM \to T_qN$ is invertible, then there is a local smooth inverse.

(i.e. $\exists$ opens $U \ni p$, $V \ni q$ and smooth $g: V \to U$ s.t. $fg = Id_V$ and $gf = Id_U$)

Main tool: Banach fixed pt theorem

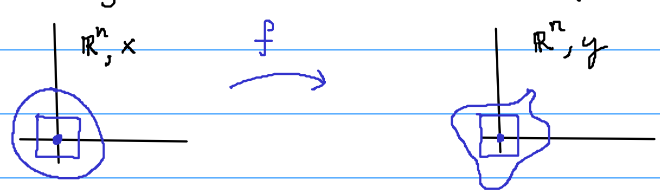
If $X \circlearrowleft h$ s.t. $d(h(x),h(y)) \leq \frac{1}{2} d(x,y)$ and X complete, $\exists!$ fixed pt.



$h:[0,1]\longrightarrow[0,1]$ contraction map

Step 0 (Setup) reduces to case M = open in $\mathbb{R}^n$, $N=\mathbb{R}^n$, $p=q=0$

also wlog $Df(0) = Id: \mathbb{R}^n \circlearrowleft$ (can replace f by $Df(0)^{-1}\circ f$)



Step 1 (Define inverse map)

Idea: for each y (suff. small) we want x s.t. $f(x)=y$ to be the fixed pt of a contraction map. (BFPT inverts f)

$$f(x) = \underset{\text{linear}}{x} + \underset{\text{nonlinear part of } f.}{k(x)} \quad \Rightarrow \quad x + k(x) = y \quad \text{as a fixed pt}$$

$$y - k(x) = x$$

for any y, define map

$$h_y : x \longmapsto y - k(x)$$

fixed pt of this map would be an inverse i.e. x st. $f(x)=y$.

Why is h_y a contraction map?

$$Dh_y(0) = 0 \Rightarrow |Dh_y| \leq \tfrac{1}{2} \text{ in some ball } B(0,r)$$

$$\text{MVT} \Rightarrow |h_y(x) - h_y(x')| \leq \tfrac{1}{2}|x - x'| \qquad \text{for } x, x' \in B(0,r)$$

Is h_y acting on a complete metric space?

$$|h_y(x)| = |h_y(x) - h_y(0) + h_y(0)| \leq |h_y(x) - h_y(0)| + |h_y(0)|$$
$$\leq \tfrac{1}{2}|x| + |y|$$

So as long as y is chosen in $B(0, \frac{r}{2})$, $\overline{B(0,r)} \xrightarrow{h_y} \overline{B(0,r)}$.

BFPT $\Rightarrow$ $\exists$ unique fixed pt of h_y in $\overline{B(0,r)}$ for each $y \in B(0, r/2)$

so we define

$$\boxed{\begin{aligned} g : B(0, r/2) &\longrightarrow \overline{B(0,r)} \\ y &\longmapsto \text{fixed pt of } h_y \end{aligned}}$$

Inverse

At this point, we know $f \circ g = Id_{B(0,r/2)}$ since $h_y(g(y)) = g(y)$

and $g \circ f = Id_{f^{-1}(B(0,r/2)) \cap \overline{B(0,r)}}$ since fixed pt in $\overline{B(0,r)}$ is unique.

but $f^{-1}(B(0,r/2)) \cap \overline{B(0,r)}$ may not be open in M ! so need to shrink $B(0, r/2)$.

Step 2 (Continuity of Inverse)

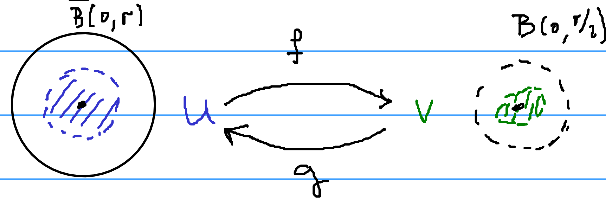
$$|g(y) - g(y')| = |h_y(g(y)) - h_{y'}(g(y'))|$$
$$\leq |y - y'| + |k(g(y)) - k(g(y'))|$$
$$\leq |y - y'| + \tfrac{1}{2}|g(y) - g(y')|$$

$\Rightarrow$ $|g(y) - g(y')| \leq 2|y - y'|$ $\Rightarrow$ g continuous.

Step 3 (f is local homeo)

$g(0) = 0$ and continuous $\Rightarrow$ let $\mathcal{U} \subset B(0,r)$ nbhd of 0 and let $V = g^{-1}(\mathcal{U})$.

then $\begin{cases} f \circ g = \mathrm{Id}_V & \text{as before} \\ g \circ f = \mathrm{Id}_{\mathcal{U}} & \text{by uniqueness of fixed pts.} \end{cases}$



Step 4 (g is differentiable at the point y):

We know what to expect the derivative to be: If g smooth,

$Dg(y)$ must be $Df(g(y))^{-1}$ by chain rule.

Now Df will be invertible on some nbhd of 0, so for this to make sense we should have chosen r small enough s.t. $Df(x)$ invertible for $x \in B(0,r)$, no problem.

Proof that Df^{-1} is the derivative: let $x = g(y)$, $x' = g(y')$.

$$|g(y) - g(y') - (Df(x))^{-1}(y-y')| = |x - x' - (Df(x))^{-1}(f(x) - f(x'))|$$
$$\leq |Df(x)|^{-1} \, |Df(x)(x-x') - (f(x) - f(x'))|$$

divide by $|y - y'|$ and note $|x - x'| \leq 2|y-y'|$ from earlier:

$$\frac{|g(y) - g(y') - (Df(x))^{-1}(y-y')|}{|y-y'|} \leq |Df(x)|^{-1} \frac{|Df(x)(x-x') - (f(x) - f(x'))|}{|x-x'|}$$

limit $y' \to y$ $\Rightarrow$ $x' \to x$ $\Longrightarrow$ RHS $\to 0$ since f diff.

$\Longrightarrow$ LHS $\to 0$ $\Rightarrow$ g differentiable at y.

Step 5 g is C^∞: $Dg(y) = Df(g(y))^{-1}$

since inversion is C^∞, g has as many derivatives as f does.