Constant Rank Theorem

Thm: If $f: M^m \to N^n$ smooth and Df has constant rank k in a nbhd of $p \in M$, then $\exists$ charts $(U,\varphi) \ni p$, $(V,\psi) \ni q$ such that

$$\psi \circ f \circ \varphi^{-1} : (x_1, \cdots, x_m) \longmapsto (x_1, \cdots, x_k, 0, \cdots, 0)$$

Step 0 Setup: choose charts so that M = open in $\mathbb{R}^m$, $p=0$, $N = \mathbb{R}^n$, $f(p)=0$.

If $Df(0)$ has rk k, this means that at least one $k \times k$ minor in matrix $Df(0)$ must be nonsingular. Reorder coordinates so it is the top left $k \times k$ minor. Label coords as follows:

$$\mathbb{R}^m \ni (\underbrace{x_1, \cdots, x_k}_{x}, \underbrace{y_1, \cdots, y_{m-k}}_{y}) \qquad (\underbrace{u_1, \cdots, u_k}_{u}, \underbrace{v_1, \cdots, v_{n-k}}_{v}) \in \mathbb{R}^n$$

With these choices, $\boxed{f(x,y) = (Q(x,y), R(x,y))}$ where

$Q = \pi_1 \circ f$, $R = \pi_2 \circ f$, $\quad \pi_1$ is proj. $(u,v) \mapsto u$

π_2 is proj $(u,v) \mapsto v$

and minor $\dfrac{\partial Q}{\partial x} = \left[\dfrac{\partial Q_i}{\partial x_j}\right]_{i,j=1\dots k}$ has nonzero determinant

$$Df = \begin{array}{c c} & \overset{\longleftarrow k \longrightarrow}{} \\ {\scriptstyle k\updownarrow} & \left[\begin{array}{c|c} \dfrac{\partial Q}{\partial x} & \dfrac{\partial Q}{\partial y} \\ \hline \dfrac{\partial R}{\partial x} & \dfrac{\partial R}{\partial y} \end{array}\right] \end{array}$$

block matrix

strategy: change coords $\phi: \mathbb{R}^m \to \mathbb{R}^m$ and $\psi: \mathbb{R}^n \to \mathbb{R}^n$ so that

$$\boxed{\psi \circ f \circ \phi^{-1} : (x,y) \longmapsto (x,0).}$$

Step 1 (Define ϕ coord change on domain which puts Q in normal form)

Def^n
$$\begin{aligned} \mathbb{R}^m &\xrightarrow{\phi} \mathbb{R}^m \\ (x,y) &\longmapsto (Q(x,y), y) \end{aligned}$$

$$D\phi = \begin{pmatrix} \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \\ 0 & 1 \end{pmatrix}$$ is invertible at 0 since $\frac{\partial Q}{\partial x}$ is.

IFT $\Rightarrow$ $\exists$ local inverse $\phi^{-1} : (x,y) \longmapsto (A(x,y), B(x,y))$.

but $(x,y) = \phi(\phi^{-1}(x,y)) = (Q(A,B), B) \Rightarrow B = y$

Then $f \circ \phi^{-1} : (x,y) \longmapsto (x, S = R(A(x,y), y))$

and $$D(f \circ \phi^{-1}) = \begin{pmatrix} I_k & 0 \\ \frac{\partial S}{\partial x} & \frac{\partial S}{\partial y} \end{pmatrix}$$

but this must still have rk k. So $\frac{\partial S}{\partial y} = 0$

$$\Rightarrow \quad f \circ \phi^{-1} : (x,y) \longmapsto (x, S(x))$$

Step 2 (Change coords on codomain to eliminate S)

$$\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

Define $$(u,v) \longmapsto (u, v - S(u))$$

this is clearly diffeo (since inverse $(u,v) \longmapsto (u, v - S(u))$)

and $$\psi \circ f \circ \phi^{-1} : (x,y) \longmapsto (x, S(x)) \longmapsto (x, 0). \qquad \square$$