3.3 Brouwer's fixed point theorem

Corollary 3.23. *Let M be a compact manifold with boundary. There is no smooth map $f : M \longrightarrow \partial M$ leaving ∂M pointwise fixed. Such a map is called a smooth retraction of M onto its boundary.*

Proof. Such a map f must have a regular value by Sard's theorem, let this value be $y \in \partial M$. Then y is obviously a regular value for $f|_{\partial M} = \mathrm{Id}$ as well, so that $f^{-1}(y)$ must be a compact 1-manifold with boundary given by $f^{-1}(y) \cap \partial M$, which is simply the point y itself. Since there is no compact 1-manifold with a single boundary point, we have a contradiction. □

For example, this shows that the identity map $S^n \to S^n$ may not be extended to a smooth map $f : \overline{B(0,1)} \to S^n$.

Lemma 3.24. *Every smooth map of the closed n-ball to itself has a fixed point.*

Proof. Let $D^n = \overline{B(0,1)}$. If $g : D^n \to D^n$ had no fixed points, then define the function $f : D^n \to S^{n-1}$ as follows: let $f(x)$ be the point in S^{n-1} nearer to x on the line joining x and $g(x)$.

This map is smooth, since $f(x) = x + tu$, where

$$u = ||x - g(x)||^{-1}(x - g(x)), \tag{70}$$

and t is the positive solution to the quadratic equation $(x+tu)\cdot(x+tu) = 1$, which has positive discriminant $b^2 - 4ac = 4(1 - |x|^2 + (x \cdot u)^2)$. Such a smooth map is therefore impossible by the previous corollary. □

Theorem 3.25 (Brouwer fixed point theorem)**.** *Any continuous self-map of D^n has a fixed point.*

Proof. The Weierstrass approximation theorem says that any continuous function on $[0, 1]$ can be uniformly approximated by a polynomial function in the supremum norm $||f||_\infty = \sup_{x\in[0,1]} |f(x)|$. In other words, the polynomials are dense in the continuous functions with respect to the supremum norm. The Stone-Weierstrass is a generalization, stating that for any compact Hausdorff space X, if A is a subalgebra of $C^0(X, \mathbb{R})$ such that A separates points $(\forall x, y, \exists f \in A \ : f(x) \neq f(y))$ and contains a nonzero constant function, then A is dense in C^0.

Given this result, approximate a given continuous self-map g of D^n by a polynomial function p' so that $||p' - g||_\infty < \epsilon$ on D^n. To ensure p' sends D^n into itself, rescale it via

$$p = (1 + \epsilon)^{-1} p'. \tag{71}$$

Then clearly p is a D^n self-map while $||p - g||_\infty < 2\epsilon$. If g had no fixed point, then $|g(x) - x|$ must have a minimum value μ on D^n, and by choosing $2\epsilon = \mu$ we guarantee that for each x,

$$|p(x) - x| \geq |g(x) - x| - |g(x) - p(x)| > \mu - \mu = 0. \tag{72}$$

Hence p has no fixed point. Such a smooth function can't exist and hence we obtain the result. □