3 Transversality

We continue to use the constant rank theorem to produce more manifolds, except now these will be cut out only locally by functions. Globally, they are cut out by intersecting with another submanifold. You should think that intersecting with a submanifold locally imposes a number of constraints equal to its codimension.

The problem is that the intersection of submanifolds need not be a submanifold; this is why the condition of transversality is so important — it guarantees that intersections are smooth.

Two subspaces $K, L \subset V$ of a vector space $V$ are transverse when $K + L = V$, i.e. every vector in $V$ may be written as a (possibly non-unique) linear combination of vectors in $K$ and $L$. In this situation one can easily see that $\dim V = \dim K + \dim L - \dim K \cap L$, or equivalently

$$\text{codim}(K \cap L) = \text{codim } K + \text{codim } L.$$  

(49)

We may apply this to submanifolds as follows:

**Definition 3.1.** Let $K, L \subset M$ be regular submanifolds such that every point $p \in K \cap L$ satisfies

$$T_pK + T_pL = T_pM.$$  

(50)

Then $K, L$ are said to be transverse submanifolds and we write $K \cap L$.

**Proposition 3.2.** If $K, L \subset M$ are transverse submanifolds, then $K \cap L$ is either empty, or a submanifold of codimension $\text{codim } K + \text{codim } L$.

**Proof.** Let $p \in K \cap L$. Then there is a neighbourhood $U$ of $p$ for which $K \cap U = f^{-1}(0)$ for $0$ a regular value of a function $f: U \to \mathbb{R}^{\text{codim } K}$ and $L \cap U = g^{-1}(0)$ for $0$ a regular value of a function $g: L \cap U \to \mathbb{R}^{\text{codim } L}$.

Then $p$ must be a regular point for $(f, g): L \cap M \cap U \to \mathbb{R}^{\text{codim } K + \text{codim } L}$, since the kernel of its derivative is the intersection $\ker Df(p) \cap \ker Dg(p)$, which is exactly $T_pK \cap T_pL$, which has codimension $\text{codim } K + \text{codim } L$ by the transversality assumption, implying $D(f, g)(p)$ is surjective. Therefore $(f, g)|_{U \cap (0, 0)} = f^{-1}(0) \cap g^{-1}(0) = K \cap L \cap U$ is a submanifold. \(\square\)

**Example 3.3** (Exotic spheres). Consider the following intersections in $\mathbb{C}^5 \setminus 0$:

$$S^7_k = \{z_1^2 + z_2^2 + z_3^4 + z_4^6 + z_5^{6k-1} = 0\} \cap \{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 = 1\}.$$  

(51)

This is a transverse intersection, and for $k = 1, \ldots, 28$ the intersection is a smooth manifold homeomorphic to $S^7$. These exotic $7$-spheres were constructed by Brieskorn and represent each of the 28 diffeomorphism classes on $S^7$.

We may choose to phrase the previous transversality result in a slightly different way, in terms of the embedding maps $k, l$ for $K, L$ in $M$. Specifically, we say the maps $k, l$ are transverse in the sense that $\forall a \in K, b \in L$ such that $k(a) = l(b) = p$, we have $\text{im}(Dk(a)) + \text{im}(Dl(b)) = T_p M$. The advantage of this approach is that it makes sense for any maps, not necessarily embeddings.
Definition 3.4. Two maps \( f : K \to M, g : L \to M \) of manifolds are called transverse when \( \text{im}(Df(a)) + \text{im}(Dg(b)) = T_pM \) for all \( a, b, p \) such that \( f(a) = g(b) = p \).

Proposition 3.5. If \( f : K \to M, g : L \to M \) are transverse smooth maps, then \( K_f \times gL = \{(a, b) \in K \times L : f(a) = g(b)\} \) is naturally a smooth manifold equipped with commuting maps

\[
\begin{array}{cccc}
K & L & \downarrow p_2 & \downarrow g \\
\uparrow i & \downarrow p_1 & \downarrow \text{id} & \downarrow f \\
K_f \times gL & L & K & M
\end{array}
\]

where \( i \) is the inclusion and \( f \cap g : (a, b) \mapsto f(a) = g(b) \).

The manifold \( K_f \times gL \) of the previous proposition is called the fiber product of \( K \) with \( L \) over \( M \), and is a generalization of the intersection product. It is often denoted simply by \( K \times_M L \), when the maps to \( M \) are clear.

Proof. Consider the graphs \( \Gamma_f \subset K \times M \) and \( \Gamma_g \subset L \times M \). To impose \( f(k) = g(l) \), we can take an intersection with the diagonal submanifold

\[
\Delta = \{(k, m, l, m) \in K \times M \times L \times M\};
\]

Step 1. We show that the intersection \( \Gamma = (\Gamma_f \times \Gamma_g) \cap \Delta \) is transverse. Let \( f(k) = g(l) = m \) so that \( x = (k, m, l, m) \in \Gamma \), and note that

\[
T_x(\Gamma_f \times \Gamma_g) = \{(v, Df(v)), (w, Dg(w)) : v \in T_kK, w \in T_lL\}
\]

whereas we also have

\[
T_x(\Delta) = \{(v, m), (w, m) : v \in T_kK, w \in T_lL, m \in T_pM\}
\]

By transversality of \( f, g \), any tangent vector \( m_i \in T_pM \) may be written as \( Df(v_i) + Dg(w_i) \) for some \( (v_i, w_i), i = 1, 2 \). In particular, we may decompose a general tangent vector to \( M \times M \) as

\[
(m_1, m_2) = (Df(v_2), Df(v_2)) + (Dg(w_1), Dg(w_1)) + (Df(v_1 - v_2), Dg(w_2 - w_1)),
\]

leading directly to the transversality of the spaces (54), (55). This shows that \( \Gamma \) is a submanifold of \( K \times M \times L \times M \).

Step 2. The projection map \( \pi : K \times M \times L \times M \to K \times L \) takes \( \Gamma \) bijectively to \( K_f \times gL \). Since (54) is a graph, it follows that \( \pi|_\Gamma : \Gamma \to K \times L \) is an injective immersion. Since the projection \( \pi \) is an open map, it also follows that \( \pi|_\Gamma \) is a homeomorphism onto its image, hence is an embedding. This shows that \( K_f \times gL \) is a submanifold of \( K \times L \).

Example 3.6. If \( K_1 = M \times Z_1 \) and \( K_2 = M \times Z_2 \), we may view both \( K_i \) as “fibering” over \( M \) with fibers \( Z_i \). If \( p_i \) are the projections to \( M \), then \( K_1 \times_M K_2 = M \times Z_1 \times Z_2 \), hence the name “fiber product.”

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Example 3.7. Let $L \subset M$ be a submanifold and let $f : K \to M$ be “transverse to $L$” in the sense that $f$ is transverse to the embedding $\iota_L : L \to M$. This means that for each pair $(k, l)$ such that $f(k) = l$, we have $Df(T_kK) + T_lL = T_lM$. Under this condition, the theorem implies that

$$f^{-1}(L) = \{ k \in K : f(k) \in L \}$$

is a smooth submanifold of $K$ (Why?) This is a generalization of the regular value theorem.

Example 3.8. Consider the Hopf map $p : S^3 \to S^2$ given by composing the embedding $S^3 \subset \mathbb{C}^2 \setminus \{0\}$ with the projection $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1 \cong S^2$. Then for any point $q \in S^2$, $p^{-1}(q) \cong S^1$. Since $p$ is a submersion, it is obviously transverse to itself, hence we may form the fiber product

$$S^3 \times_{S^2} S^3,$$

which is a smooth 4-manifold equipped with a map $p \cap p$ to $S^2$ with fibers $(p \cap p)^{-1}(q) \cong S^1 \times S^1$.

These are our first examples of nontrivial fiber bundles, which we shall explore later.