3 Transversality

We continue to use the constant rank theorem to produce more manifolds, except now these will be cut out only *locally* by functions. Globally, they are cut out by intersecting with another submanifold. You should think that intersecting with a submanifold locally imposes a number of constraints equal to its codimension.

The problem is that the intersection of submanifolds need not be a submanifold; this is why the condition of transversality is so important - it guarantees that intersections are smooth.

Two subspaces $K, L \subset V$ of a vector space V are *transverse* when $K + L = V$, i.e. every vector in V may be written as a (possibly non-unique) linear combination of vectors in K and L. In this situation one can easily see that $\dim V = \dim K + \dim L - \dim K \cap L$, or equivalently

$$\text{codim}(K \cap L) = \text{codim}K + \text{codim}L. \tag{49}$$

We may apply this to submanifolds as follows:

Definition 3.1. Let $K, L \subset M$ be regular submanifolds such that every point $p \in K \cap L$ satisfies

$$T_pK + T_pL = T_pM. \tag{50}$$

Then K, L are said to be *transverse* submanifolds and we write $K \pitchfork L$.

Proposition 3.2. *If $K, L \subset M$ are transverse submanifolds, then $K \cap L$ is either empty, or a submanifold of codimension* $\text{codim}K + \text{codim}L$.

Proof. Let $p \in K \cap L$. Then there is a neighbourhood U of p for which $K \cap U = f^{-1}(0)$ for 0 a regular value of a function $f : U \longrightarrow \mathbb{R}^{\text{codim}K}$ and $L \cap U = g^{-1}(0)$ for 0 a regular value of a function $g : L \cap U \longrightarrow \mathbb{R}^{\text{codim}L}$.

Then p must be a regular point for $(f, g) : L \cap M \cap U \longrightarrow \mathbb{R}^{\text{codim}K+\text{codim}L}$, since the kernel of its derivative is the intersection $\ker Df(p) \cap \ker Dg(p)$, which is exactly $T_pK \cap T_pL$, which has codimension $\text{codim}K + \text{codim}L$ by the transversality assumption, implying $D(f, g)(p)$ is surjective. Therefore $(f, g)|_{\tilde{U}}^{-1}(0, 0) = f^{-1}(0) \cap g^{-1}(0) = K \cap L \cap \tilde{U}$ is a submanifold. ☐

Example 3.3 (Exotic spheres)**.** Consider the following intersections in $\mathbb{C}^5 \backslash 0$:

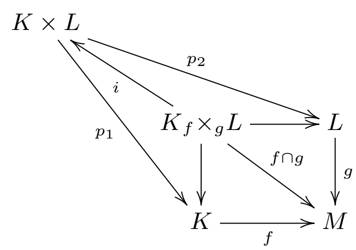
$$S_k^7 = \{z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} = 0\} \cap \{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 = 1\}. \tag{51}$$

This is a transverse intersection, and for $k = 1, \ldots, 28$ the intersection is a smooth manifold homeomorphic to S^7. These exotic 7-spheres were constructed by Brieskorn and represent each of the 28 diffeomorphism classes on S^7.

We may choose to phrase the previous transversality result in a slightly different way, in terms of the embedding maps k, l for K, L in M. Specifically, we say the maps k, l are transverse in the sense that $\forall a \in K, b \in L$ such that $k(a) = l(b) = p$, we have $\text{im}(Dk(a)) + \text{im}(Dl(b)) = T_pM$. The advantage of this approach is that it makes sense for any maps, not necessarily embeddings.

Definition 3.4. Two maps $f : K \longrightarrow M$, $g : L \longrightarrow M$ of manifolds are called *transverse* when $\mathrm{im}(Df(a)) + \mathrm{im}(Dg(b)) = T_pM$ for all a, b, p such that $f(a) = g(b) = p$.

Proposition 3.5. *If $f : K \longrightarrow M$, $g : L \longrightarrow M$ are transverse smooth maps, then $K{}_f{\times}_g L = \{(a,b) \in K \times L \ : \ f(a) = g(b)\}$ is naturally a smooth manifold equipped with commuting maps*



(52)

where i is the inclusion and $f \cap g : (a,b) \mapsto f(a) = g(b)$.

The manifold $K{}_f{\times}_g L$ of the previous proposition is called the *fiber product* of K with L over M, and is a generalization of the intersection product. It is often denoted simply by $K \times_M L$, when the maps to M are clear.

Proof. Consider the graphs $\Gamma_f \subset K \times M$ and $\Gamma_g \subset L \times M$. To impose $f(k) = g(l)$, we can take an intersection with the diagonal submanifold

$$\Delta = \{(k, m, l, m) \in K \times M \times L \times M\}. \tag{53}$$

Step 1. We show that the intersection $\Gamma = (\Gamma_f \times \Gamma_g) \cap \Delta$ is transverse. Let $f(k) = g(l) = m$ so that $x = (k, m, l, m) \in \Gamma$, and note that

$$T_x(\Gamma_f \times \Gamma_g) = \{((v, Df(v)), (w, Dg(w))),\ v \in T_kK, \quad w \in T_lL\} \tag{54}$$

whereas we also have

$$T_x(\Delta) = \{((v, m), (w, m)) \ : \ v \in T_kK, \quad w \in T_lL, \quad m \in T_pM\} \tag{55}$$

By transversality of f, g, any tangent vector $m_i \in T_pM$ may be written as $Df(v_i) + Dg(w_i)$ for some (v_i, w_i), $i = 1, 2$. In particular, we may decompose a general tangent vector to $M \times M$ as

$$(m_1, m_2) = (Df(v_2), Df(v_2)) + (Dg(w_1), Dg(w_1)) + (Df(v_1 - v_2), Dg(w_2 - w_1)), \tag{56}$$

leading directly to the transversality of the spaces (54), (55). This shows that Γ is a submanifold of $K \times M \times L \times M$.

Step 2. The projection map $\pi : K \times M \times L \times M \to K \times L$ takes Γ bijectively to $K{}_f{\times}_g L$. Since (54) is a graph, it follows that $\pi|_\Gamma : \Gamma \to K \times L$ is an injective immersion. Since the projection π is an open map, it also follows that $\pi|_\Gamma$ is a homeomorphism onto its image, hence is an embedding. This shows that $K{}_f{\times}_g L$ is a submanifold of $K \times L$. □

Example 3.6. If $K_1 = M \times Z_1$ and $K_2 = M \times Z_2$, we may view both K_i as "fibering" over M with fibers Z_i. If p_i are the projections to M, then $K_1 \times_M K_2 = M \times Z_1 \times Z_2$, hence the name "fiber product".

Example 3.7. Let $L \subset M$ be a submanifold and let $f : K \to M$ be "transverse to L" in the sense that f is transverse to the embedding $\iota_L : L \to M$. This means that for each pair (k, l) such that $f(k) = l$, we have $Df(T_k K) + T_l L = T_l M$. Under this condition, the theorem implies that

$$f^{-1}(L) = \{k \in K \ : \ f(k) \in L\}$$

is a smooth submanifold of K (Why?) This is a generalization of the regular value theorem.

Example 3.8. Consider the Hopf map $p : S^3 \longrightarrow S^2$ given by composing the embedding $S^3 \subset \mathbb{C}^2 \backslash \{0\}$ with the projection $\pi : \mathbb{C}^2 \backslash \{0\} \longrightarrow \mathbb{C}P^1 \cong S^2$. Then for any point $q \in S^2$, $p^{-1}(q) \cong S^1$. Since p is a submersion, it is obviously transverse to itself, hence we may form the fiber product

$$S^3 \times_{S^2} S^3,$$

which is a smooth 4-manifold equipped with a map $p \cap p$ to S^2 with fibers $(p \cap p)^{-1}(q) \cong S^1 \times S^1$.

These are our first examples of nontrivial fiber bundles, which we shall explore later.