

2.5 Local structure of smooth maps

In some ways, smooth manifolds are easier to produce or find than general topological manifolds, because of the fact that smooth maps have linear approximations. Therefore smooth maps often behave like linear maps of vector spaces, and we may gain inspiration from vector space constructions (e.g. subspace, kernel, image, cokernel) to produce new examples of manifolds.

In charts (U, φ) , (V, ψ) for the smooth manifolds M, N , a smooth map $f : M \rightarrow N$ is represented by a smooth map $\psi \circ f \circ \varphi^{-1} \in C^\infty(\varphi(U), \mathbb{R}^n)$. We shall give a general local classification of such maps, based on the behaviour of the derivative. The fundamental result which provides information about the map based on its derivative is the *inverse function theorem*.

Theorem 2.15 (Inverse function theorem). *Let $f : (M, p) \rightarrow (N, q)$ be a smooth map of n -dimensional manifolds and suppose that $Df(p) : T_p M \rightarrow T_q N$ is invertible. Then f has a local smooth inverse. That is, there are neighbourhoods U, V of p, q and a smooth map $g : V \rightarrow U$ such that $f \circ g = \text{id}_V$ and $g \circ f = \text{id}_U$.*

Proof. Without loss of generality, we can take M to be a neighbourhood of the origin in \mathbb{R}^n and $N = \mathbb{R}^n$, and assume that $f(0) = 0$. We can also assume $Df(0) = \text{Id}$, since we can replace f by $(Df(0))^{-1} \circ f$ (linear change of variables). We are trying to invert f , so solve the equation $y = f(x)$ uniquely for x . Define k so that $f(x) = x + k(x)$. Hence $k(x)$ is the nonlinear part of f .

The claim is that if y is in a sufficiently small neighbourhood of the origin, then the map $h_y : x \mapsto y - k(x)$ is a contraction mapping on some closed ball; it then has a unique fixed point $g(y)$, and so $y - k(g(y)) = g(y)$, i.e. g is an inverse for f .

Why is h_y a contraction mapping? Note that $Dh_y(0) = 0$ and hence there is a ball $B(0, r)$ where $\|Dh_y\| \leq \frac{1}{2}$. This then implies (mean value theorem) that for $x, x' \in B(0, r)$,

$$\|h_y(x) - h_y(x')\| \leq \frac{1}{2}\|x - x'\|.$$

Therefore h_y does look like a contraction, we just have to make sure it's operating on a complete metric space. Let's estimate the size of $h_y(x)$:

$$\|h_y(x)\| \leq \|h_y(x) - h_y(0)\| + \|h_y(0)\| \leq \frac{1}{2}\|x\| + \|y\|.$$

Therefore by taking $y \in B(0, \frac{r}{2})$, the map h_y is a contraction mapping on $\overline{B(0, r)}$. Let $g(y)$ be the unique fixed point of h_y guaranteed by the contraction mapping theorem.

To see that ϕ is continuous (and hence f is a homeomorphism), we compute

$$\begin{aligned} \|g(y) - g(y')\| &= \|h_y(g(y)) - h_{y'}(g(y'))\| \\ &\leq \|h_y(g(y)) - h_y(g(y'))\| + \|y - y'\| \\ &\leq \frac{1}{2}\|g(y) - g(y')\| + \|y - y'\|, \end{aligned}$$

so that we have $\|g(y) - g(y')\| \leq 2\|y - y'\|$, as required.

Having shown that g is continuous, we can choose an open set $U \subset B(0, r)$ and define $V = g^{-1}(U) \subset B(0, \frac{r}{2})$. Then $f \circ g = \text{id}_V$ by the fixed point property and $g \circ f = \text{id}_U$ by the uniqueness of fixed points in the closed ball, proving that $f : U \rightarrow V$ is indeed a homeomorphism.

To see that g is differentiable, we guess the derivative $(Df)^{-1}$ and compute. Let $x = g(y)$ and $x' = g(y')$. For this to make sense we must have chosen r small enough so that Df is nonsingular on $B(0, r)$, which is not a problem.

$$\begin{aligned} \|g(y) - g(y') - (Df(x))^{-1}(y - y')\| &= \|x - x' - (Df(x))^{-1}(f(x) - f(x'))\| \\ &\leq \|(Df(x))^{-1}\| \|(Df(x))(x - x') - (f(x) - f(x'))\|. \end{aligned}$$

Now note that $\|(Df(x))^{-1}\|$ is bounded and $\|x - x'\| \leq 2\|y - y'\|$ as shown before. Dividing by $\|y - y'\|$, taking the limit $y \rightarrow y'$, and using the differentiability of f , we get that g is differentiable, and with derivative $(Df)^{-1}$. That is,

$$Dg = (Df)^{-1}. \quad (41)$$

Since inversion is C^∞ , g has as many derivatives as f , hence g is C^∞ . \square

This theorem provides us with a local normal form for a smooth map with $Df(p)$ invertible: we may choose coordinates on sufficiently small neighbourhoods of $p, f(p)$ so that f is represented by the identity map $\mathbb{R}^n \rightarrow \mathbb{R}^n$.

In fact, the inverse function theorem leads to a normal form theorem for a more general class of maps:

Theorem 2.16 (Constant rank theorem). *Let $f : M^m \rightarrow N^n$ be a smooth map such that Df has constant rank k in a neighbourhood of $p \in M$. Then there are charts (U, φ) and (V, ψ) containing $p, f(p)$ such that*

$$\psi \circ f \circ \varphi^{-1} : (x_1, \dots, x_m) \mapsto (x_1, \dots, x_k, 0, \dots, 0). \quad (42)$$

Proof. Begin by choosing charts so that without loss of generality M is an open set in \mathbb{R}^m and N is \mathbb{R}^n .

Since $\text{rk } Df = k$ at p , there is a $k \times k$ minor of $Df(p)$ with nonzero determinant. Reorder the coordinates on \mathbb{R}^m and \mathbb{R}^n so that this minor is top left, and translate coordinates so that $f(0) = 0$. label the coordinates $(x_1, \dots, x_k, y_1, \dots, y_{m-k})$ on the domain and $(u_1, \dots, u_k, v_1, \dots, v_{n-k})$ on the codomain.

Then we may write $f(x, y) = (Q(x, y), R(x, y))$, where Q is the projection to $u = (u_1, \dots, u_k)$ and R is the projection to v . with $\frac{\partial Q}{\partial x}$ nonsingular. First we wish to put Q into normal form. Consider the map $\phi(x, y) = (Q(x, y), y)$, which has derivative

$$D\phi = \begin{pmatrix} \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \\ 0 & 1 \end{pmatrix} \quad (43)$$

As a result we see $D\phi(0)$ is nonsingular and hence there exists a local inverse $\phi^{-1}(x, y) = (A(x, y), B(x, y))$. Since it's an inverse this means $(x, y) = \phi(\phi^{-1}(x, y)) = (Q(A, B), B)$, which implies that $B(x, y) = y$.

Then $f \circ \phi^{-1} : (x, y) \mapsto (x, S = R(A, y))$, and must still be of rank k . Since its derivative is

$$D(f \circ \phi^{-1}) = \begin{pmatrix} I_{k \times k} & 0 \\ \frac{\partial S}{\partial x} & \frac{\partial S}{\partial y} \end{pmatrix} \quad (44)$$

we conclude that $\frac{\partial S}{\partial y} = 0$, meaning that we have eliminated the y -dependence:

$$f \circ \phi^{-1} : (x, y) \mapsto (x, S(x)). \quad (45)$$

We now postcompose by the diffeomorphism $\sigma : (u, v) \mapsto (u, v - S(u))$, to obtain

$$\sigma \circ f \circ \phi^{-1} : (x, y) \mapsto (x, 0), \quad (46)$$

as required. \square

As we shall see, these theorems have many uses. One of the most straightforward uses is for defining submanifolds.

There are several ways to define the notion of submanifold. We will use a definition which works for topological and smooth manifolds, based on the local model of inclusion of a vector subspace. These are sometimes called *regular* or *embedded* submanifolds.

Definition 2.17. A subspace $L \subset M$ of an m -manifold is called a submanifold of codimension k when each point $x \in L$ is contained in a chart (U, φ) for M such that

$$L \cap U = f^{-1}(0), \quad (47)$$

where f is the composition of φ with the projection $\mathbb{R}^m \rightarrow \mathbb{R}^k$ to the last k coordinates (x_{m-k+1}, \dots, x_m) . A submanifold of codimension 1 is usually called a *hypersurface*.

Proposition 2.18. *If $f : M \rightarrow N$ is a smooth map of manifolds, and if $Df(p)$ has constant rank on M , then for any $q \in f(M)$, the inverse image $f^{-1}(q) \subset M$ is a regular submanifold.*

Proof. Let $x \in f^{-1}(q)$. Then there exist charts ψ, φ such that $\psi \circ f \circ \varphi^{-1} : (x_1, \dots, x_m) \mapsto (x_1, \dots, x_k, 0, \dots, 0)$ and $f^{-1}(q) \cap U = \{x_1 = \dots = x_k = 0\}$. Hence we obtain that $f^{-1}(q)$ is a codimension k submanifold. \square

Example 2.19. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be given by $(x_1, \dots, x_n) \mapsto \sum x_i^2$. Then $Df(x) = (2x_1, \dots, 2x_n)$, which has rank 1 at all points in $\mathbb{R}^n \setminus \{0\}$. Hence since $f^{-1}(q)$ contains $\{0\}$ iff $q = 0$, we see that $f^{-1}(q)$ is a regular submanifold for all $q \neq 0$. Exercise: show that this manifold structure is compatible with that obtained in Example 1.22.

The previous example leads to the following special case.

Proposition 2.20. *If $f : M \rightarrow N$ is a smooth map of manifolds and $Df(p)$ has rank equal to $\dim N$ along $f^{-1}(q)$, then this subset $f^{-1}(q)$ is an embedded submanifold of M .*

Proof. Since the rank is maximal along $f^{-1}(q)$, it must be maximal in an open neighbourhood $U \subset M$ containing $f^{-1}(q)$, and hence $f : U \rightarrow N$ is of constant rank. \square

Definition 2.21. If $f : M \rightarrow N$ is a smooth map such that $Df(p)$ is surjective, then p is called a *regular point*. Otherwise p is called a *critical point*. If all points in the level set $f^{-1}(q)$ are regular points, then q is called a *regular value*, otherwise q is called a critical value. In particular, if $f^{-1}(q) = \emptyset$, then q is regular.

It is often useful to highlight two classes of smooth maps; those for which Df is everywhere *injective*, or, on the other hand *surjective*.

Definition 2.22. A smooth map $f : M \rightarrow N$ is called a *submersion* when $Df(p)$ is surjective at all points $p \in M$, and is called an *immersion* when $Df(p)$ is injective at all points $p \in M$. If f is an injective immersion which is a homeomorphism onto its image (when the image is equipped with subspace topology), then we call f an *embedding*.

Proposition 2.23. *If $f : M \rightarrow N$ is an embedding, then $f(M)$ is a regular submanifold.*

Proof. Let $f : M \rightarrow N$ be an embedding. Then for all $m \in M$, we have charts $(U, \varphi), (V, \psi)$ where $\psi \circ f \circ \varphi^{-1} : (x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, 0, \dots, 0)$. If $f(U) = f(M) \cap V$, we're done. To make sure that some other piece of M doesn't get sent into the neighbourhood, use the fact that $f(U)$ is open in the subspace topology. This means we can find a smaller open set $V' \subset V$ such that $V' \cap f(M) = f(U)$. Restricting the coordinates to V' , we see that $f(M)$ is cut out by (x_{m+1}, \dots, x_n) , where $n = \dim N$. \square

Example 2.24. If $\iota : M \rightarrow N$ is an embedding of M into N , then $D\iota : TM \rightarrow TN$ is also an embedding (hence so are $D^k\iota : T^kM \rightarrow T^kN$), showing that TM is a submanifold of TN .

2.6 Smooth maps between manifolds with boundary

We may also use the constant rank theorem to study manifolds with boundary.

Proposition 2.25. *Let M be a smooth n -manifold and $f : M \rightarrow \mathbb{R}$ a smooth and proper real-valued function, and let a, b , with $a < b$, be regular values of f . Then $f^{-1}([a, b])$ is a cobordism between the closed $n-1$ -manifolds $f^{-1}(a)$ and $f^{-1}(b)$.*

Proof. The pre-image $f^{-1}((a, b))$ is an open subset of M and hence a submanifold. Since p is regular for all $p \in f^{-1}(a)$, we may (by the constant rank theorem) find charts such that f is given near p by the linear map

$$(x_1, \dots, x_m) \mapsto x_m. \tag{48}$$

Possibly replacing x_m by $-x_m$, we therefore obtain a chart near p for $f^{-1}([a, b])$ into H^m , as required. Proceed similarly for $p \in f^{-1}(b)$. \square

Example 2.26. Using $f : \mathbb{R}^n \rightarrow \mathbb{R}$ given by $(x_1, \dots, x_n) \mapsto \sum x_i^2$, this gives a simple proof for the fact that the closed unit ball $\overline{B(0, 1)} = f^{-1}([-1, 1])$ is a manifold with boundary.

Example 2.27. Consider the C^∞ function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ given by $(x, y, z) \mapsto x^2 + y^2 - z^2$. Both $+1$ and -1 are regular values for this map, with pre-images given by 1- and 2-sheeted hyperboloids, respectively. Hence $f^{-1}([-1, 1])$ is a cobordism between hyperboloids of 1 and 2 sheets. In other words, it defines a cobordism between the disjoint union of two closed disks and the closed cylinder (each of which has boundary $S^1 \sqcup S^1$). Does this cobordism tell us something about the cobordism class of a connected sum?

Proposition 2.28. *Let $f : M \rightarrow N$ be a smooth map from a manifold with boundary to the manifold N . Suppose that $q \in N$ is a regular value of f and also of $f|_{\partial M}$. Then the pre-image $f^{-1}(q)$ is a submanifold with boundary². Furthermore, the boundary of $f^{-1}(q)$ is simply its intersection with ∂M .*

Proof. If $p \in f^{-1}(q)$ is not in ∂M , then as before $f^{-1}(q)$ is a submanifold in a neighbourhood of p . Therefore suppose $p \in \partial M \cap f^{-1}(q)$. Pick charts φ, ψ so that $\varphi(p) = 0$ and $\psi(q) = 0$, and $\psi \circ f \circ \varphi^{-1}$ is a map $U \subset H^m \rightarrow \mathbb{R}^n$. Extend this to a smooth function \tilde{f} defined in an open set $\tilde{U} \subset \mathbb{R}^m$ containing U . Shrinking \tilde{U} if necessary, we may assume \tilde{f} is regular on \tilde{U} . Hence $\tilde{f}^{-1}(0)$ is a submanifold of \mathbb{R}^m of codimension n .

Now consider the real-valued function $\pi : \tilde{f}^{-1}(0) \rightarrow \mathbb{R}$ given by the restriction of $(x_1, \dots, x_m) \mapsto x_m$. $0 \in \mathbb{R}$ must be a regular value of π , since if not, then the tangent space to $\tilde{f}^{-1}(0)$ at 0 would lie completely in $x_m = 0$, which contradicts the fact that q is a regular point for $f|_{\partial M}$.

Hence, by Proposition 2.25, we have expressed $f^{-1}(q)$, in a neighbourhood of p , as a regular submanifold with boundary given by $\{\varphi^{-1}(x) : x \in \tilde{f}^{-1}(0) \text{ and } \pi(x) \geq 0\}$, as required. \square

²i.e. locally modeled on the inclusion $H^k \subset H^n$ given by $(x_1, \dots, x_k) \mapsto (0, \dots, 0, x_1, \dots, x_k)$.