2.4 Flow of a vector field

A smooth curve in the manifold $M$ is by definition a smooth map from $\mathbb{R}$ to $M$

$$\gamma : \mathbb{R} \to M.$$  

The domain $\mathbb{R}$ has a natural coordinate $t$, and a natural coordinate vector field $\hat{\gamma}$, and if we apply the derivative of $\gamma$ to this vector field, we get the velocity of the path, defined as follows:

$$\dot{\gamma}(t) = (D\gamma)|_t(\frac{\partial}{\partial t}).$$

The velocity is therefore a path in $TM$ which “lifts the path $\gamma$”, in the sense that the following diagram commutes:

$$\begin{array}{ccc}
TM & \xrightarrow{\pi} & M \\
\gamma \downarrow & & \downarrow \\
\mathbb{R} & \xrightarrow{\gamma} & M
\end{array}$$

Given a vector field $X \in \mathfrak{X}(M)$ and an initial point $x \in M$, there is a natural dynamical system, where $x$ is made to evolve in time according to the rule that its velocity at all times must coincide with the vector field $X$. This idea is captured in the following precise way.

**Definition 2.9.** The smooth curve $\gamma$ is called an integral curve of the vector field $X \in \mathfrak{X}(M)$ when its velocity is $X$, that is,

$$\dot{\gamma}(t) = X(\gamma(t)). \quad (40)$$

If we choose a coordinate chart $(U, \Psi)$ for $M$ containing the path $\gamma$, we may write $\gamma$ in components: $\Psi \circ \gamma$ is nothing but an $n$-tuple of functions $(\gamma^1, \ldots, \gamma^n)$ of one variable $t$. Also, using the chart we may write the vector field $X$ in components, giving a vector-valued function of $n$ variables

$$(X_1(x^1, \ldots, x^n), \ldots, X_n(x^1, \ldots, x^n)).$$

Then the integral curve equation (40), written in components, states that

$$\frac{d}{dt}(\gamma^i) = X_i(\gamma^1, \ldots, \gamma^n), \quad i = 1, \ldots, n.$$

This is a system of ordinary differential equations, and so the existence and uniqueness theorem for ODE guarantees that it has a unique solution on some time interval $(-\epsilon, \epsilon)$, $\epsilon > 0$, once an initial point $(\gamma^1(0), \ldots, \gamma^n(0))$ is chosen. This tells us that integral curves $\gamma$ always exist and are unique in a neighbourhood of zero once we fix $\gamma(0)$. In fact, the theorem also guarantees that the integral curve depends smoothly on the initial condition.

We may state the theorem from ODE as follows:

**Theorem 2.10** (Existence and uniqueness theorem for ODE). Let $X$ be a vector field defined on an open set $V \subset \mathbb{R}^n$. For each point $x_0 \in V$ there exists a neighbourhood $U$ of $x_0$ in $V$, a number $\epsilon > 0$, and a smooth map

$$\Phi : (-\epsilon, \epsilon) \times U \to V$$

$$(t, x) \mapsto \varphi_t(x),$$

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such that for all \( x \in U \), the curve \( t \mapsto \varphi_t(x) \) is an integral curve of \( X \) with initial condition \( \varphi_0(x) = x \). Furthermore, if \( (U', \mathcal{E}') \) is another tuple satisfying the same conditions, then \( \Phi \) coincides with \( \Phi' \) on \( (-\tau, \tau) \times (U \cap U') \), where \( \tau = \min(\epsilon, \epsilon') \).

**Corollary 2.11.** Let \( X \in \mathcal{X}(M) \). There exists an open neighbourhood \( U \) of \( \{0\} \times M \) in \( \mathbb{R} \times M \) and a smooth map \( \Phi: U \to M \) such that, for each \( x \in M \), we have

1) \( (\mathbb{R} \times \{x\}) \cap U \) is an interval about zero;
2) \( t \mapsto \varphi_t(y) = \Phi(t, y) \) is an integral curve of \( X \);
3) \( \varphi_0(y) = y \);
4) if \( (t, x), (t + t', x), (t', \varphi_t(x)) \) are all in \( U \) then \( \varphi_{t+t'}(\varphi_t(x)) = \varphi_{t+t'}(x) \).

Furthermore, if \( (U', \Phi') \) is as above and satisfies 1), 2), 3), then it must satisfy iv), and \( \Phi = \Phi' \) on \( U \cap U' \).

**Proof.** Using the previous theorem, we can find an open cover \( (U_i)_{i \in I} \) of \( M \) and a sequence \( (\epsilon_i)_{i \in I}, \epsilon_i > 0 \), and maps \( \Phi_i : (-\epsilon_i, \epsilon_i) \times U_i \to M \) with the properties given in the theorem. By the uniqueness given in the theorem, \( \Phi_i \) coincides with \( \Phi_j \) on the intersection of their respective domains, and so we obtain a well-defined map

\[
\Phi : U = \bigcup_{i \in I} ((-\epsilon_i, \epsilon_i) \times U_i) \to M.
\]

By construction, \( \Phi \) satisfies properties 1), 2), 3). To verify property iv), notice that \( \tau \mapsto \varphi_\tau(\varphi_t(x)) \) and \( \tau \mapsto \varphi_{t+t'}(x) \), for \( 0 \leq \tau \leq t' \), are both integral curves for \( X \) with initial condition \( \varphi_t(x) \), and so must coincide, in particular the coincide for \( \tau = t' \). The final uniqueness statement is proven exactly in the same way.

Such data \( (U, \Phi) \) is sometimes called the *flow* of the vector field \( X \). More precisely, it is called a local 1-parameter group of diffeomorphisms generated by \( X \), for the simple reason that if \( W \subset M \) is an open set such that \( \{t\} \times W \) and \( \{-t\} \times \varphi_t(W) \) are contained in \( U \), then \( \varphi_t : W \to \varphi_t(W) \) is a diffeomorphism with inverse \( \varphi_{-t} \). Furthermore, if \( \{t'\} \times \varphi_t(W) \) and \( \{t + t'\} \times W \) are contained in \( U \), then we have the composition law

\[
\varphi_{t'} \circ \varphi_t = \varphi_{t+t'}, \quad \text{or} \quad e^nX \circ e^n'X = e^{(t+t')}X,
\]

if we use the exponential notation \( \varphi_t = e^{tX} \) to emphasize this group structure. Note that this is an intrinsic family of diffeomorphisms associated to \( X \), and does not coincide with the Riemannian exponential map in Riemannian geometry, which uses the geodesic flow.

If the domain \( U \) is actually the whole of \( \mathbb{R} \times M \), then we call this structure a global 1-parameter group of diffeomorphisms. Note that, due to the uniqueness in Corollary 2.11, we may take the union of all possible domains of local 1-parameter groups of diffeomorphisms generated by \( X \); this is the unique maximal local 1-parameter group of diffeomorphisms generated by \( X \).
Definition 2.12. The vector field $X$ is complete when it generates a
1-parameter group of diffeomorphisms. That is, its flow is defined
for all time.

Theorem 2.13. Any vector field on a compact manifold is complete.

Proof. Let $(U, \Phi)$ be the maximal local 1-parameter group of diffeomor-
phisms generated by $X$. For a contradiction, suppose that $x \in M$ is
such that $U \cap (\mathbb{R} \times \{x\})$ is an open interval with finite upper limit $\omega$ (the
lower limit case is done similarly). Now using compactness, let $y$ be an
accumulation point for $\Phi(t, x)$ as $t$ approaches $\omega$. We may then use the
flow defined near $y$ to extend $\Phi(t, x)$ as follows, which contradicts the
maximality of $\Phi$:

Let $\delta > 0$ and a neighbourhood $W$ of $y$ be sufficiently small that
$(-\delta, \delta) \times W \subset U$, and let $\tau \in (\omega - \delta, \omega)$ be such that $\varphi_{\tau}(x) \in W$. Then
we can find a neighbourhood $V$ of $x$ with the property that $\tau \times V \subset U$
and $\varphi_{\tau}(V) \subset W$. Then if we enlarge $U$ to $U \cup ((\omega - \delta, \omega + \delta) \times V)$, we
we can extend $\Phi$ by

$$
\Phi'(t, x) = \Phi(t - \tau, \Phi(\tau, x)), \quad \text{for } (t, x) \in (\omega - \delta, \omega + \delta) \times V.
$$

Example 2.14. The vector field $X = x^2 \frac{\partial}{\partial x}$ on $\mathbb{R}$ is not complete. For
initial condition $x_0$, have integral curve $\gamma(t) = x_0(1 - tx_0)^{-1}$, which gives
$\Phi(t, x_0) = x_0(1 - tx_0)^{-1}$, which is well-defined on

$$
U = \{1 - tx > 0\} \subset \mathbb{R} \times \mathbb{R}.
$$