2.4 Flow of a vector field

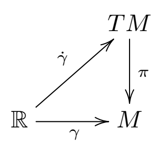
A smooth curve in the manifold M is by definition a smooth map from $\mathbb{R}$ to M

$$\gamma : \mathbb{R} \to M.$$

The domain $\mathbb{R}$ has a natural coordinate t, and a natural coordinate vector field $\frac{\partial}{\partial t}$, and if we apply the derivative of γ to this vector field, we get the velocity of the path, defined as follows:

$$\dot{\gamma}(t) = (D\gamma)|_t(\tfrac{\partial}{\partial t}).$$

The velocity is therefore a path in TM which "lifts the path γ", in the sense that the following diagram commutes:



Given a vector field $X \in \mathfrak{X}(M)$ and an initial point $x \in M$, there is a natural *dynamical system*, where x is made to evolve in time according to the rule that its velocity at all times must coincide with the vector field X. This idea is captured in the following precise way.

Definition 2.9. The smooth curve γ is called an *integral curve* of the vector field $X \in \mathfrak{X}(M)$ when its velocity is X, that is,

$$\dot{\gamma}(t) = X(\gamma(t)). \tag{40}$$

If we choose a coordinate chart (U, Ψ) for M containing the path γ, we may write γ in components: $\Psi \circ \gamma$ is nothing but an n-tuple of functions $(\gamma^1, \dots, \gamma^n)$ of one variable t. Also, using the chart we may write the vector field X in components, giving a vector-valued function of n variables

$$(X_1(x^1, \dots, x^n), \dots, X_n(x^1, \dots, x^n)).$$

Then the integral curve equation (40), written in components, states that

$$\tfrac{d}{dt}(\gamma^i) = X_i(\gamma^1, \dots \gamma^n), \qquad i = 1, \dots, n.$$

This is a system of ordinary differential equations, and so the existence and uniqueness theorem for ODE guarantees that it has a unique solution on some time interval $(-\epsilon, \epsilon)$, $\epsilon > 0$, once an initial point $(\gamma^1(0), \dots, \gamma^n(0))$ is chosen. This tells us that integral curves γ always exist and are unique in a neighbourhood of zero once we fix $\gamma(0)$. In fact, the theorem also guarantees that the integral curve depends smoothly on the initial condition. We may state the theorem from ODE as follows:

Theorem 2.10 (Existence and uniqueness theorem for ODE)**.** *Let X be a vector field defined on an open set $V \subset \mathbb{R}^n$. For each point $x_0 \in V$ there exists a neighbourhood U of x_0 in V, a number $\epsilon > 0$, and a smooth map*

$$\begin{aligned} \Phi : (-\epsilon, \epsilon) \times U &\to V \\ (t, x) &\mapsto \varphi_t(x), \end{aligned}$$

such that for all $x \in U$, the curve $t \mapsto \varphi_t(x)$ is an integral curve of X with initial condition $\varphi_0(x) = x$. Furthermore, if (U', ϵ', Φ') is another tuple satisfying the same conditions, then Φ coincides with Φ' on $(-\tau, \tau) \times (U \cap U')$, where $\tau = \min(\epsilon, \epsilon')$.

Corollary 2.11. *Let $X \in \mathfrak{X}(M)$. There exists an open neighbourhood U of $\{0\} \times M$ in $\mathbb{R} \times M$ and a smooth map $\Phi : U \to M$ such that, for each $x \in M$, we have*

- *i)* $(\mathbb{R} \times \{x\}) \cap U$ *is an interval about zero;*
- *ii)* $t \mapsto \varphi_t(y) = \Phi(t, y)$ *is an integral curve of X;*
- *iii)* $\varphi_0(y) = y$*;*
- *iv) if* $(t, x), (t+t', x), (t', \varphi_t(x))$ *are all in U then* $\varphi_{t'}(\varphi_t(x)) = \varphi_{t+t'}(x)$.

Furthermore, if (U', Φ') is as above and satisfies $i), ii), iii)$, then it must satisfy $iv)$, and $\Phi = \Phi'$ on $U \cap U'$.

Proof. Using the previous theorem, we can find an open cover $(U_i)_{i\in I}$ of M and a sequence $(\epsilon_i)_{i\in I}$, $\epsilon_i > 0$, and maps $\Phi_i : (-\epsilon_i, \epsilon_i) \times U_i \to M$ with the properties given in the theorem. By the uniqueness given in the theorem, Φ_i coincides with Φ_j on the intersection of their respective domains, and so we obtain a well-defined map

$$\Phi : U = \bigcup_{i\in I} ((-\epsilon_i, \epsilon_i) \times U_i) \to M.$$

By construction, Φ satisfies properties $i), ii), iii)$. To verify property $iv)$, notice that $\tau \mapsto \varphi_\tau(\varphi_t(x))$ and $\tau \mapsto \varphi_{t+\tau}(x)$, for $0 \leq \tau \leq t'$, are both integral curves for X with initial condition $\varphi_t(x)$, and so must coincide, in particular the coincide for $\tau = t'$. The final uniqueness statement is proven exactly in the same way. □

Such data (U, Φ) is sometimes called the *flow* of the vector field X. More precisely, it is called a *local 1-parameter group of diffeomorphisms* generated by X, for the simple reason that if $W \subset M$ is an open set such that $\{t\} \times W$ and $\{-t\} \times \varphi_t(W)$ are contained in U, then $\varphi_t : W \to \varphi_t(W)$ is a diffeomorphism with inverse φ_{-t}. Furthermore, if $\{t'\} \times \varphi_t(W)$ and $\{t + t'\} \times W$ are contained in U, then we have the composition law

$$\varphi_{t'} \circ \varphi_t = \varphi_{t'+t}, \quad \text{or} \quad e^{tX} \circ e^{t'X} = e^{(t+t')X},$$

if we use the exponential notation $\varphi_t = e^{tX}$ to emphasize this group structure. Note that this is an intrinsic family of diffeomorphisms associated to X, and does not coincide with the *Riemannian exponential map* in Riemannian geometry, which uses the geodesic flow.

If the domain U is actually the whole of $\mathbb{R} \times M$, then we call this structure a *global 1-parameter group of diffeomorphisms*. Note that, due to the uniqueness in Corollary 2.11, we may take the union of all possible domains of local 1-parameter groups of diffeomorphisms generated by X; this is the unique maximal local 1-parameter group of diffeomorphisms generated by X.

Definition 2.12. The vector field X is *complete* when it generates a global 1-parameter group of diffeomorphisms. That is, its flow is defined for all time.

Theorem 2.13. *Any vector field on a compact manifold is complete.*

Proof. Let (U, Φ) be the maximal local 1-parameter group of diffeomorphisms generated by X. For a contradiction, suppose that $x \in M$ is such that $U \cap (\mathbb{R} \times \{x\})$ is an open interval with finite upper limit ω (the lower limit case is done similarly). Now using compactness, let y be an accumulation point for $\Phi(t, x)$ as t approaches ω. We may then use the flow defined near y to extend $\Phi(t, x)$ as follows, which contradicts the maximality of Φ:

Let $\delta > 0$ and a neighbourhood W of y be sufficiently small that $(-\delta, \delta) \times W \subset U$, and let $\tau \in (\omega - \delta, \omega)$ be such that $\varphi_\tau(x) \in W$. Then we can find a neighbourhood V of x with the property that $\{\tau\} \times V \subset U$ and $\varphi_\tau(V) \subset W$. Then if we enlarge U to $U \cup ((\omega - \delta, \omega + \delta) \times V)$, we can extend Φ by

$$\Phi'(t, x) = \Phi(t - \tau, \Phi(\tau, x)), \quad \text{for } (t, x) \in (\omega - \delta, \omega + \delta) \times V.$$

□

Example 2.14. The vector field $X = x^2 \frac{\partial}{\partial x}$ on $\mathbb{R}$ is not complete. For initial condition x_0, have integral curve $\gamma(t) = x_0(1 - tx_0)^{-1}$, which gives $\Phi(t, x_0) = x_0(1 - tx_0)^{-1}$, which is well-defined on

$$U = \{1 - tx > 0\} \subset \mathbb{R} \times \mathbb{R}.$$