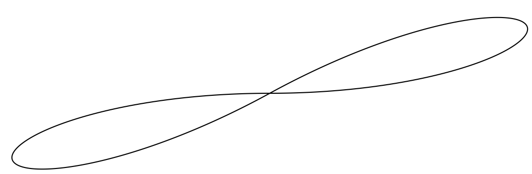
Exercise 1. Let X be compact and $f : X \longrightarrow Y$ smooth with $\dim X = \dim Y$ and Y connected. Recall that the mod 2 degree of f is defined in terms of the mod 2 intersection number as follows: $\deg_2(f) = I_2(f, \iota)$, where $\iota : y \mapsto Y$ is the inclusion map of a point $y \in Y$.

1. Prove that $\deg_2(f)$ is independent of the point $y \in Y$.
2. A map $f : X \longrightarrow Y$ is called *essential* when it is not homotopic to a constant map. Prove that if $\deg_2(f) = 1$ and $\dim Y > 0$, then f is essential .
3. Can there exist a smooth map $f : S^2 \longrightarrow T^2$ with $\deg_2(f) = 1$? [Hint: consider two embedded circles C_1, C_2 in T^2 intersecting transversally at a single point.] Can there exist a smooth map of $\deg_2(f) = 1$ in the opposite direction? In each case, give proofs.

Exercise 2. Let $f : S^1 \longrightarrow \mathbb{R}^2$ be an embedding and choose $p \in \mathbb{R}^2 \backslash f(S^1)$. Define $f_p : S^1 \longrightarrow S^1$ by $f_p(z) = \frac{f(z)-p}{|f(z)-p|}$. Then we define the mod 2 winding number of f about p to be the degree of f_p, i.e. $w_2(f, p) = \deg_2(f_p)$.

1. Compute $w_2(f, p)$ for the standard embedding of S^1 in $\mathbb{R}^2$, and for any p.
2. Let $R_p(v)$ be the ray starting at p with direction $v \in S^1$. Prove that $v \in S^1$ is a critical value of f_p if and only if $R_p(v)$ is somewhere tangent to $f(S^1)$.
3. Show that $w_2(f, p)$ coincides with the number of points mod 2 in $R_p(v) \cap f(S^1)$, whenever v is a regular value of f_p.
4. Show that there are points $p, q \in \mathbb{R}^2 \backslash f(S^1)$ such that $w_2(f, p) = 0$ and $w_2(f, q) = 1$. Show that this implies that $\mathbb{R}^2 \backslash f(S^1)$ has at least two components.

Exercise 3. Consider an immersion $i : S^1 \to \mathbb{R}^2$ following a "figure eight path" as shown below.



Prove that there is no smooth homotopy from i to the standard embedding $j : S^1 \to \mathbb{R}^2$ which remains an immersion at all intermediate times.

Exercise 4. Let $f : M \to \mathbb{R}$ be a proper submersion. Then $V = \ker Df$ defines a codimension 1 subbundle of TM called the vertical bundle.

1. Show, using a partition of unity, that it is possible to choose a rank 1 subbundle $H \subset TM$ complementary to V. Do not use a Riemannian metric.
2. Conclude that to any vector field v on $\mathbb{R}$ we may associate a unique vector field v^h on M which lies in H. This is called the horizontal lift of v.
3. Prove the preimages of any pair of points in the image of f are diffeomorphic manifolds.