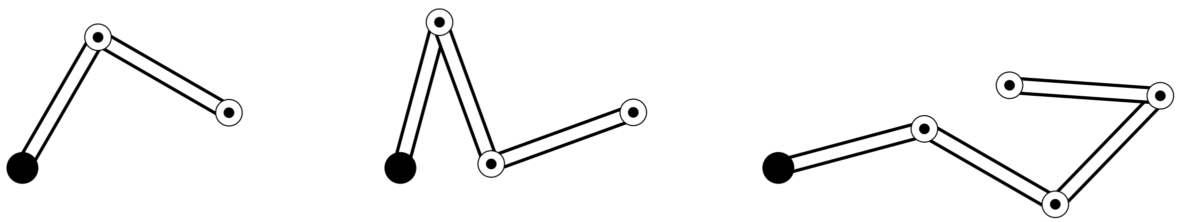
Exercise 1. Fix a positive integer k and complex numbers $c_0, c_1, \ldots, c_{k-1}$. Consider the map $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = z^k + c_{k-1}z^{k-1} + \cdots + c_1 z + c_0.$$

1. Identifying $\mathbb{C}$ with the standard affine chart U_0 for $\mathbb{C}P^1$, prove that f extends uniquely to a smooth map from $\mathbb{C}P^1$ to $\mathbb{C}P^1$. Call this map $\tilde{f}$.

2. Describe in detail the critical points and critical values of $\tilde{f}$ in the case $k = 2$.

Exercise 2. Let K be a manifold with boundary where L, M are without boundary. Assume that $f : K \to M$ and $g : L \to M$ are smooth maps such that both f and ∂f are transverse to g. Prove that the fiber product $K \times_M L$ is a manifold with boundary equal to $\partial K \times_M L$.



Exercise 3. A planar arm consists of a number n of rigid unit length segments in a plane, joined end-to-end, starting with an end with fixed position and ending with a free end, as shown above.

1. Give a bijection from the space of configurations X of such a planar arm with n segments to $T^n = (S^1)^n$.

2. If the fixed end is located at $(0, 0) \in \mathbb{R}^2$, define the map $f : X \to \mathbb{R}^2$ on any configuration by taking the position of its free end. Describe precisely the critical points and critical values of this map. Give examples in the cases $n = 2, 3, 4$. (Using $\mathbb{C}$ instead of $\mathbb{R}$ might be preferred by some, this is fine)

3. Consider the configurations $Y \subset X$ where the free end is constrained to lie on the positive x-axis $\{(x, y) \ : \ x > 0, y = 0\}$. Prove that Y is a smooth submanifold.

4. On this submanifold $Y \subset X$, consider the function $g : Y \to \mathbb{R}$ given by the distance of the free end from the fixed end. Describe carefully the critical points and critical values of this map in the cases $n = 2, 3, 4$.

5. In the case $n = 3$, prove that $g^{-1}(2)$ is diffeomorphic to S^1.

6. In the case $n = 4$, prove that $g^{-1}(3)$ is diffeomorphic to S^2. Challenge: what is $g^{-1}(1)$?

Bonus 3.1. Let K, L be submanifolds of a manifold M, and suppose that their intersection $K \cap L$ is also a submanifold. Then K, L are said to have *clean* intersection when, for each $p \in K \cap L$, we have $T_p(K \cap L) = T_pK \cap T_pL$. Show that there are coordinates near $p \in K \cap L$ such that K, L, and $K \cap L$ are given by linear subspaces of $\mathbb{R}^n$ of the form $V(x^{i_1}, \ldots, x^{i_k})$ for some subset of the coordinates. It is useful to use the algebraic geometry notation $V(x^1, \ldots, x^k)$ to mean the "vanishing" subspace $x^1 = \cdots = x^k = 0$.