Exercise 1. Construct, using the stereographic charts for S^2 given in class, a smooth vector field on S^2 which vanishes exactly at 2 points, and another vector field which vanishes at exactly 1 point.

Exercise 2. Determine whether or not the vector field $(1+x^2)\frac{\partial}{\partial x}$ is complete on $\mathbb{R}$.

Exercise 3. Let $M = \mathbb{R}^n$, with coordinates $(x^1, \ldots, x^n)$. Then $TM = \mathbb{R}^n \times \mathbb{R}^n$.

1. The trivial map $E : M \to TM$ given by $x \mapsto (x, x)$ defines a section of the tangent bundle, i.e. a vector field. Write the vector field in the given coordinates. Compute the time-t flow of this vector field and determine whether it is complete or not. Draw a picture of the vector field in the cases $n = 1$ and $n = 2$.

2. Suppose $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map with matrix coefficients a_{ij}. Then the map $A : x \mapsto (x, Ax)$ defines a vector field on M; Write the vector field in coordinates, compute its flow and determine if it is complete.

Exercise 4. Prove that the orthogonal group $O(n,\mathbb{R}) = \{X \in GL(n,\mathbb{R}) : XX^\top = 1\}$ is a smooth submanifold of $M(n,\mathbb{R})$, the $n \times n$ matrices. To show this, consider the map $f : M(n,\mathbb{R}) \to S(n,\mathbb{R})$ to the symmetric matrices $S(n,\mathbb{R})$ given by $f(X) = XX^\top$.

Exercise 5. Prove that if K is a submanifold of L and L is a submanifold of M, then K is a submanifold of M.

Exercise 6.

1. Let $\varphi : \mathbb{R}P^2 \to \mathbb{R}^3$ be the map defined by

$$\varphi([x : y : z]) = \frac{1}{x^2 + y^2 + z^2}(yz, xz, xy).$$

   Show that φ is smooth. Is it an immersion? If not, at which points in the domain does φ fail to be an immersion?

2. Let $\Phi : \mathbb{R}P^2 \to \mathbb{R}^4$ be the map

$$\Phi([x : y : z]) = \frac{1}{x^2 + y^2 + z^2}(x^2 - z^2, yz, xz, xy).$$

   Show that this is an embedding of $\mathbb{R}P^2$ into $\mathbb{R}^4$.