1.7 Examples of handle decompositions

Example 1.19. Consider $S^2$ embedded in $\mathbb{R}^3$ as an irregular ellipsoid $x^2 + 2y^2 + 3z^2 = 1$. The “height function” $f = r^2$ is Morse, and invariant under the antipodal map $A : (x, y, z) \mapsto (-x, -y, -z)$. Hence it descends to a Morse function with three critical points on $\mathbb{R}P^2$, with indices 0, 1, 2. In this case we can see explicitly how the handle decomposition works; after the 1-handle is attached to the 0-handle, the manifold is non-orientable, so that it has one boundary component and so is ready for the single 2-handle.

Example 1.20 (Planar equilateral pentagons). A planar equilateral pentagon can be described by 4 unit complex numbers $(z_1, z_2, z_3, z_4) = (e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4})$ with $\text{Im} \sum z_i = 0$ and $\text{Re} \sum z_i = 1$. We will analyze this space by viewing it as the -1 level set of the function $f = -\text{Re} \sum z_i = \sum \cos \theta_i$. Note that $-1$ is a regular value since $df = [\sin \theta_i]$ only vanishes when $\theta_i \in \pi\mathbb{Z}$, so that $\cos \theta_i = \pm 1$, so that $\text{Re} \sum z_i = 0 \pmod{2}$.

The space $f < 0$ is indeed a smooth 3-dimensional manifold embedded in $T^4$, and we use $f$ as a Morse function on it. The minimum value is at $-4$, where there is a single critical point of index 0. Then there is another critical value at $f = -2$, with the following critical points $(-1, -1, -1, 1), (-1, -1, 1, -1), (-1, 1, -1, -1)$ and $(1, -1, -1, -1)$. These are all of index 1, since “the function can only be decreased in one direction”. There are no other critical values in $(-\infty, 1]$. Hence we see that the 3-manifold may be constructed by 4 1-handle attachments on a 3-ball. This means the boundary must be a genus 4 orientable (it is boundary of an orientable 3-manifold) Riemann surface.

1.8 Morse inequalities

In this section we compare the Morse polynomial of a Morse function $f$ on the manifold $M$

$$\mathcal{M}_t(f) = \sum_{p \in \text{Crit}(f)} t^\lambda = \sum_\lambda \mu_f(\lambda) t^\lambda,$$

with the Poincaré polynomial $P_t(M) = \sum b_i t^i$, where $b_i = \dim_k H_i(M, k)$ are the Betti numbers of $M$ with respect to some coefficient field $k$.

To analyze the difference between these polynomials, let us do so “inductively” on the sublevel sets $M^a = f^{-1}(\infty, a]$. Let the Morse polynomial for $M^a$ be $\mathcal{M}_t(f)^a$.

If there are no critical points in $[a, b]$, then clearly $\mathcal{M}_t(f)^a = \mathcal{M}_t(f)^b$ by definition and $P_t(M^a) = P_t(M^b)$, by theorem A.

If there is a single critical point of index $\lambda$ in $[a, b]$, then by definition

$$\mathcal{M}_t(f)^b - \mathcal{M}_t(f)^a = t^\lambda.$$

What happens to the Poincaré polynomial? We use Theorem B, which states that that $M^b$ is a $\lambda$-handle attached to $M^a$.

Consider the attaching sphere $S^{\lambda - 1} \subset M^a$. This cycle either bounds a chain in $M^a$ or not. This is a global criterion in $M^a$.

Completable case: if the attaching sphere bounds a chain in $M^a$: then this chain, together with the new $\lambda$-handle, forms a new nontrivial cycle of dimension $\lambda$. Then $\Delta P_t = t^\lambda$, and therefore

$$\Delta(\mathcal{M}_t - P_t) = 0.$$

If the attaching sphere is a nontrivial cycle in $M^a$: then the new $\lambda$-handle kills this cycle and we get $\Delta P_t = t^{\lambda - 1}$, and so

$$\Delta(\mathcal{M}_t - P_t) = t^\lambda + t^{\lambda - 1} = t^{\lambda - 1}(1 + t).$$

By induction, therefore, we have
Theorem 1.21 (Morse inequalities).

\[ \mathcal{M}_t(f) - P_t(M) = (1 + t)Q_t(f), \]

where \( Q_t(f) \) is a polynomial with nonnegative integer coefficients. In particular we have

- \( \mu_r(\lambda) \geq b_0 \) (Weak Morse inequalities),
- \( \chi(M) = \sum (-1)^i \mu(i) \),
- for each \( k \geq 0 \),
  \[ b_k - b_{k-1} + \cdots + (-1)^k b_0 \leq \mu_k - \mu_{k-1} + \cdots + (-1)^k \mu_0. \]

Proof. The weak Morse inequalities are obtained by simply truncating the equation \( \mathcal{M}_t - P_t = (1 + t)Q_t \).

To give a rigorous proof of the main statement, we will use the exact sequence in relative homology for the inclusion \( M^a \subset M^b \). First we determine \( H_a(M^b, M^a, k) \) using excision:

Excision says that if \( Z \subset A \subset X \) with \( Z \subset A^{\text{int}} \) then the inclusion \( (X - Z, A - Z) \subset (X, A) \) induces an isomorphism on homology. For \( (M^b, M^a) \), we can take \( Z \) to be the complement in \( M^a \) of a small tubular neighbourhood of the attaching sphere. Then \( H_n(M^b, M^a; k) = H_n(\lambda, S_1(k); \tilde{M}) \), which is \( k \) for \( n = \lambda \) and zero otherwise. By the long exact sequence in relative homology therefore we have

\[ 0 \rightarrow H_{\lambda}(M^a) \rightarrow H_{\lambda}(M^b) \rightarrow \mathbb{K} \rightarrow H_{\lambda-1}(M^a) \rightarrow H_{\lambda-1}(M^b) \rightarrow 0 \]

Whether \( \delta = 0 \) or rank 1 gives the two alternatives: if \( \delta = 0 \) then \( \Delta P = t^\lambda \) and we are in the completable case.

Corollary 1.22. \( \chi(M) = 0 \) for \( M \) an odd-dimensional compact manifold.

Proof. If \( f \) is Morse, so that \( \chi(M) = \mathcal{M}_1(f) \), then \( -f \) is Morse also, so \( \chi(M) = \mathcal{M}_1(-f) = -\mathcal{M}_1(f) = -\chi(M) \).

Definition 10. \( f \) is a perfect Morse function for the coefficient field \( k \) if \( \mathcal{M}_t(f) = P_t(M, k) \).

Example 1.23. Note that \( \mathbb{R}P^n \) has \( H_k(\mathbb{R}P^n, \mathbb{R}) \) vanishes except for \( k = 0 \) and, when \( n \) odd, \( k = n \). Applying the Morse inequalities we get that there is at least one critical point. However, over \( \mathbb{K} = \mathbb{Z}_2 \), we have \( H_k(\mathbb{R}P^n) = \mathbb{Z}_2 \) for all \( 0 \leq k \leq n \). Hence the Morse inequalities yield at least \( n + 1 \) critical points. This bound is achieved by the function (a generalization of the above ellipsoid)

\[ f = \sum_{i=1}^{n+1} i |x_i|^2 \]

Corollary 1.24. Let \( M \) be a compact manifold. If the gap condition \( |\lambda(p) - \lambda(q)| \neq 1 \) holds for all \( p, q \in \text{Crit}(f) \), then \( f \) is a perfect Morse function (for any field).

Proof. Under the gap assumption we wish to show that the connecting homomorphism \( \delta : H_{\lambda}(M^b, M^a) \rightarrow H_{\lambda-1}(M^a) \) is always zero. Assuming that \( H_{\lambda}(M^b, M^a) \) is not itself zero, this means by the gap that \( \lambda - 1 \) is not a morse index for the manifold. Assuming inductively that this means \( H_{\lambda-1}(M^a) = 0 \), this implies that \( \delta \) must vanish. But then we obtain the exact sequence

\[ 0 \rightarrow H_k(M^a) \rightarrow H_k(M^b) \rightarrow H_k(M^b, M^a) \rightarrow 0 \]

so that if \( k \) is not a morse index, then \( H_k(M^a) = 0 \) implies \( H_k(M^a) = 0 \) as well, establishing the induction.

Example 1.25. The height function on the sphere has Morse polynomial \( 1 + t^n \), which satisfies the gap condition for \( n > 1 \), and so gives the Betti numbers of \( S^n \). For \( |m - n| \geq 2 \), the gap condition is satisfied for the sum of height functions on \( S^m \times S^n \), so that \( 1 + t^m + t^n + t^{m+n} \) gives the Betti numbers for \( S^m \times S^n \). Finally, the Morse function defined earlier on \( \mathbb{C}P^n \) had all even indices. Hence it satisfies the gap condition and is perfect.