Under assumptions on $X$ (connected, local simple-connected, and semi-locally simply connected, in order to define the topology of $\tilde{X}$) we constructed a universal covering $(\tilde{X}, p)$, by setting

$$\tilde{X} = \{ [\gamma] : \gamma \text{ is a path in } X \text{ starting at } x_0 \}.$$ 

We also saw that this space has trivial fundamental group, as follows: Any path $\gamma$ in $X$ may be lifted to $\tilde{X}$ by defining $\tilde{\gamma}(t)$ to be the path $\gamma$ up to time $t$ (and constant afterwards). If $[\gamma]$ is in the image of $p_*$, this means that there is a loop in this class, say $\gamma$, which lifts to a loop $\tilde{\gamma}$ in $\tilde{X}$. But this means that $\gamma$ up to time 1 is equal in $\tilde{X}$ (i.e. homotopic to) to $\gamma$ up to time 0, i.e. $[\gamma] = 0$ in $\pi_1(X)$. Since $p_*$ is injective, it must be that $\pi_1(\tilde{X}) = 0$.

Having the universal cover, we can produce all other coverings via quotients of it, as follows:

**surjectivity of functor.** Suppose now that $(X, x_0)$ has a (path-connected) universal covering space $(\tilde{X}, \tilde{x}_0)$, and suppose a subgroup $H \subset \pi_1(X, x_0)$ is specified. Then we define an equivalence relation on $\tilde{X}$ as follows: given points $[\gamma], [\gamma'] \in \tilde{X}$, we define $[\gamma] \sim [\gamma']$ iff $\gamma(1) = \gamma'(1)$ and $[\gamma\gamma'^{-1}] \in H$. Because $H$ is a subgroup, this is an equivalence relation. Now set $X_H = \tilde{X}/\sim$. Note that this equivalence relation holds for nearby paths in the sense $[\gamma] \sim [\gamma']$ iff $[\gamma\eta] \sim [\gamma'\eta]$. Therefore, if any two points in $U[\gamma], U[\gamma']$ are equivalent, then so is every other point in the neighbourhood. This shows that the projection $p : X_H \longrightarrow X$ via $[\gamma] \mapsto \gamma(1)$ is a covering map.

As a basepoint in $X_H$, pick $[x_0]$, the constant path at $x_0$. Then the image of $p_*$ is $H$, since the lift of the loop $\gamma$ is a path beginning at $[x_0]$ and ending at $[\gamma]$, and this is a loop exactly when $[\gamma] \sim [x_0]$, i.e. $[\gamma] \in H$.

**Example 1.37** (Diagram: page 58). Consider the wedge $S^1 \vee S^1$. Recall that $\pi_1(S^1 \vee S^1) = F_2 = \langle a, b \rangle$. View it as a graph with one vertex and two edges, labeled by $a, b$ with their appropriate orientations. We can then take any other graph $\tilde{X}$, labeled in this way, and such that each vertex is locally isomorphic to the given vertex, and define a covering map to $S^1 \vee S^1$. The resulting graph $\tilde{X}$ is itself a wedge of k circles, with fundamental group $F_k$. Hence we obtain a map $F_k \longrightarrow F_2$ which is an injection. Examples (1), (2)

In fact, every 4-valent graph can be labeled in the way required above: if the graph is finite, take an Eulerian circuit and label the edges $a, b, a, b, \ldots$. Then the a edges are a collection of disjoint circles: orient them and do the same for the b edges.

An infinite 4-valent graph may be constructed which is a simply-connected covering space for $S^1 \vee S^1$: it is a fractal 4-branched tree (drawing).

Not only can we have a free group on any number of generators as a subgroup of $F_2$, but also we can have infinitely many generators (drawing of (10), (11))

Note that changing the basepoint vertex of a covering simply conjugates $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ inside $\pi_1(X, x_0)$. (draw (3), (4)). Isomorphism of coverings (without fixing basepoints) is just a graph isomorphism preserving labeling and orientation.

Note also that characteristic subgroups may be isomorphic without being conjugate. (draw (5), (6)), these are homeomorphic graphs, but not isomorphic as covering spaces.

**Example 1.38.** If $X$ is a path-connected space with fundamental group $\pi_1(X, x_0)$, then by attaching 2-cells $e^2_\alpha$ via maps $\varphi_\alpha : S^1 \longrightarrow X$, then the resulting space $Y$ will have fundamental group which is a quotient of $\pi_1(X, x_0)$ by the normal subgroup $N$ generated by loops of the form $\gamma_\alpha \varphi_\alpha \gamma_\alpha^{-1}$, for any $\gamma_\alpha$ chosen to join $x_0$ to $\varphi_\alpha(1)$. This is seen by Van Kampen’s theorem applied to a thickened version $Z$ of $Y$ where the paths $\gamma_\alpha$ are thicken to intervals attached to the discs $e_\alpha$.

We can use this construction to obtain any group as a fundamental group. Choose a presentation

$$G = \langle g_\alpha \mid r_\beta \rangle.$$ 

This is possible since any group is a quotient of a free group. Then we construct $X_G$ from $\vee_\alpha S^1 \partial_\alpha$ by attaching 2-cells $e^2_\beta$ by loops specified by the words $r_\beta$. (for example, to obtain $Z_n = \langle a \mid a^n = 1 \rangle$, attach a single 2-cell to $S^1$ via the map $z \mapsto z^n$. For $n = 2$ we obtain $\mathbb{RP}^2$. 

15
The Cayley complex is one way of describing the universal cover of $X_G$. It is a cell complex $\tilde{X}_G$ constructed as follows: The vertices are the elements of $G$ itself. Then at each vertex $g \in G$, attach an edge joining $g$ to $gg_a$ for each generator $g_a$. The resulting graph is the Cayley graph of $G$ with respect to the generators $g_a$. Then, each relation $r_\beta$ determines a loop starting at any $g \in G$, and we attach a 2-cell to all these loops. There is an obvious map to $X_G$ given by quotienting by the action of $G$ on the left, which sends all points to the basepoint, each edge $g \rightarrow gg_a$ to the edge $S^1_\alpha$, and each 2-cell associated to $r_\beta$ to that attached in the construction of $X_G$.

For example, consider $G = \mathbb{Z}_2 \ast \mathbb{Z}_2 = \langle a, b \mid a^2 = b^2 = 1 \rangle$. Then the Cayley graph has vertices $\{\ldots, bab, ba, b, e, a, ab, aba, \ldots\}$, and two generators so there will be four edges coming in/out of each vertex $g$: two outward edges corresponding to right multiplication by $a, b$ to $ga, gb$, and two inward coming from $ga^{-1}, gb^{-1}$. We therefore obtain an infinite sequence of tangent circles. We produce the Cayley complex by attaching a 2-cell corresponding to $a^2$ to the loop produced at each vertex $g$ by following the loop $g \rightarrow ga \rightarrow ga^2$. This attaches two 2-cells to each circle, yielding a sequence of tangent 2-spheres, clearly a simply-connected space. The action of $G$ corresponds to an action by even translations $(ab)$ and the antipodal maps, giving the quotient space $\mathbb{R}P^2 \vee \mathbb{R}P^2$. 
1.7 Group actions and Deck transformations

In many cases we obtain covering spaces \( \tilde{X} \to X \) from group actions; if a group \( A \) acts on \( \tilde{X} \), the quotient map \( \tilde{X} \to X \) may, under some assumptions on \( A \) and its action, be a covering.

For example, we can define the \( n \)-fold covering \( S^1 \to S^1 \) as simply the quotient of \( S^1 \) by the action of \( Z_n \) via \( x \mapsto x e^{2\pi i/n} \) for \( a = e^{2\pi i/n} \), or even \( \mathbb{R} \to S^1 \) via the quotient by the \( Z \) action \( x \mapsto x + n \).

In general, if \( p : (\tilde{X}, \tilde{x}_0) \to (X, x_0) \) is a universal cover, then we can obtain \( X \) as a quotient of \( \tilde{X} \) by the action of the fundamental group \( \pi_1(X, x_0) \) as follows:

Given an element \( [\gamma] \in \pi_1(X, x_0) \), \( \gamma \) lifts to a path terminating in \( \tilde{x}_0' \) over \( x_0 \). Now the covering \( p \) has a unique lift to \( \tilde{X} \), sending \( \tilde{x}_0 \) to the alternative basepoint \( \tilde{x}_0' \). This lift is a homeomorphism \( \tilde{X} \to \tilde{X} \), and this defines an action of \( \pi_1(X, x_0) \) on \( \tilde{X} \). We’ll be careful in a moment to show the quotient is \( X \).

In general, not all covering maps \( p \) will be the quotient by the action of a group: this will only be the case for normal covering maps, i.e. those for which \( p_* (\pi_1(\tilde{X})) \) is a normal subgroup \( N \); Then \( \pi_1(X, x_0)/N \) is a group, and this will act in the same way as above, with quotient \( X \).

**Example 1.39** (Coverings of surfaces). There are many interesting coverings of surfaces, which can be constructed by acting by symmetry groups:

An example of a covering of a compact surface: take a genus \( mn+1 \) surface, draw it as a surface with \( m \) genus \( n \) legs and a hole in the center. There is an obvious \( \mathbb{Z}_n \) symmetry by rotating by \( 2\pi/m \). The quotient map is then a \( m \)-fold covering map to a surface of genus \( n+1 \).

Consider a genus \( g \) surface in \( \mathbb{R}^3 \) with the holes along an axis, and consider the rotation about this axis by \( \pi \), giving a \( \mathbb{Z}_2 \) action with \( 2(g+1) \) fixed points. Remove the fixed points. The punctured surface then has a 2-sheeted cover of \( S^2 \) punctured in \( 2(g+1) \) points. This is the topological description of an equation \( y^2 = f(z) \) with \( f \) of degree \( 2g+1 \) (this way, \( y^2 = f \) has exactly two solutions except at the \( 2g+1 \) zeros of \( f \) and the point at infinity where \( f = \infty \). The particular case where \( f \) has degree 3 defines a genus 1 surface, which is called an elliptic curve once a complex structure is chosen on it.

**Example 1.40.** The antipodal map on \( S^n \) is an action of \( \mathbb{Z}_2 \) with no fixed points; the quotient map is a covering of \( \mathbb{R}P^n \). This will imply that \( \pi_1(\mathbb{R}P^n) = \mathbb{Z}_2 \). In the case \( n = 3 \), this 2:1 cover is also known as the sequence of groups:

\[
0 \to \mathbb{Z}_2 = \{\pm 1\} \to SU(2) \to SO(3) \to 0
\]

Note that \( SO(3) \) has several famous finite subgroups: the cyclic groups \( A_n \), the dihedral groups \( D_n \), and the symmetry groups of the tetrahedron, and dodecahedron, \( E_6, E_7, E_8 \). In this way we can construct other covering spaces, e.g. \( S^3 \to S^3/\pi^{-1}(E_8) \), the Poincaré dodecahedral space, a homology 3-sphere.

To formalize the observations above, we wish to answer the following questions: Given a connected covering space (without basepoint), what is its group of automorphisms (deck transformations), and when does this group define the covering as a quotient? And, more generally, when is a group action defining a covering map?

**Definition 10.** A covering map \( p : \tilde{X} \to X \) is called normal when, for each \( x \in X \) and each pair of lifts \( \tilde{x}, \tilde{x}' \) of \( x \), there is an automorphism of \( p \) taking \( \tilde{x} \) to \( \tilde{x}' \).

**Theorem 1.41.** If \( p : \tilde{X} \to X \) is a path-connected covering (of \( X \) path-connected and locally path-connected), with characteristic subgroup \( H \), then the group of automorphisms of \( p \) is \( A = N(H)/H \), and the quotient \( \tilde{X}/A \) is the covering with characteristic subgroup \( N(H) \). Therefore, a covering is normal precisely when \( H \) is normal, and in this case the automorphism group is \( A = \pi_1(X)/H \) and \( \tilde{X}/A = X \).

**Proof.** Changing the basepoint from \( \tilde{x}_0 \in p^{-1}(x_0) \) to \( \tilde{x}_1 \in p^{-1}(x_0) \) corresponds to conjugating \( H \) by \( [\gamma] \in \pi_1(X, x_0) \) which lifts to a path \( \tilde{\gamma} \) from \( \tilde{x}_0 \) to \( \tilde{x}_1 \). Therefore, \( [\gamma] \in N(H) \) iff \( p_* (\pi_1(\tilde{X}, \tilde{x}_0)) = p_* (\pi_1(\tilde{X}, \tilde{x}_1)) \), which is the case (by the lifting of maps) iff there is a deck transformation taking \( \tilde{x}_0 \) to \( \tilde{x}_1 \). Therefore \( \tilde{X} \) is normal iff \( N(H) = \pi_1(X, x_0) \), i.e. \( H \) is already normal in \( \pi_1(X, x_0) \).

In general there is a group homomorphism \( \varphi : N(H) \to A \), sending \( [\gamma] \) to the deck transformation mapping \( \tilde{x}_0 \mapsto \tilde{x}_1 \) as above. It is surjective by the argument above, and its kernel is precisely the classes \( [\gamma] \) lifting to loops, i.e. the elements of \( H \) itself.
Theorem 1.42. Suppose $G$ acts on $Y$ in a properly discontinuous way, i.e. each $y \in Y$ has a neighbourhood $U$ such that $gU$ are disjoint for all $g \in G$. Then the quotient of $Y$ by $G$ is a normal covering map, and if $Y$ is path-connected then $G$ is the automorphism group of the cover.

Proof. First we remark that deck transformations of a covering space obviously have the properly discontinuous property.

To prove the result, take any open set $U$ as in the definition of proper discontinuity. Then the quotient map identifies the disjoint homeomorphic neighbourhoods \( \{ g(U) : g \in G \} \) with $p(U) \subset Y/G$. By the definition of the quotient topology, this gives a homeomorphism on each component, and hence we have a covering.

Certainly $G$ is a subgroup of the deck transformations, and the covering space is normal since $g_2g_1^{-1}$ takes $g_1(U)$ to $g_2(U)$, and if $Y$ is path-connected then $G$ equals the deck transformations, since if a deck transformation $f$ sends $y$ to $f(y)$, we may simply lift the covering to the alternative point $f(y)$ (the lifting criterion is satisfied since the cover is normal) and this deck transformation must coincide with $f$ by uniqueness.

Remark 1. Suppose $p: \tilde{X} \to X$ is a finite covering. Fixing $x_0 \in X$, we have two natural permutation actions on the finite set $p^{-1}(x_0)$: one is by $\pi_1(X, x_0)$, via lifting of loops, i.e. given $[\gamma] \in \pi_1(X, x_0)$, the permutation $\sigma([\gamma])$ acts on $\tilde{x}_0$ by $\sigma(\tilde{x}_0) = \tilde{\gamma}(1)$, where $\tilde{\gamma}$ is the lift of $\gamma$ starting at $\tilde{x}_0$. The study of this permutation action is an alternative approach to classifying covering spaces, and this is described in Hatcher. It is useful to understand both approaches.

The second action is by the group of deck transformations $A = N(H)/H$ (for the characteristic subgroup $H$). These actions commute. Interestingly, when $\tilde{X}$ is the universal cover, $A$ is $\pi_1(X, x_0)$ as well, and so we have the same group acting in two ways- these actions need not coincide.