You may only quote theorems we have covered in the class so far.

Question 1. Let $X = S^1 \sqcup S^1$ be the disjoint union of two circles and let $T^2 = S^1 \times S^1$ be the 2-torus. Give examples of:

i) a smooth embedding $f : X \longrightarrow T^2$,

ii) a smooth immersion $f : X \longrightarrow T^2$ which fails to be an embedding,

iii) a smooth map $f : X \longrightarrow T^2$ which is not an immersion.

iv) a smooth map $f : X \longrightarrow T^2$ which is not an embedding but for which $f(X)$ is a regular submanifold diffeomorphic to X.

v) a smooth map $f : S^2 \longrightarrow T^2$ (from the 2-sphere S^2) which is not a constant map.

Question 2. Give an example of a compact, connected 3-manifold X with boundary such that X is not diffeomorphic to the closed unit 3-ball, yet ∂X is diffeomorphic to S^2.

Also, give examples of compact, connected 3-manifolds with boundary X_1, X_2 such that $\partial X_1 = S^2 \sqcup T^2$ and $\partial X_2 = S^2 \sqcup S^2 \sqcup S^2$.

Question 3. Let X be compact and $f : X \longrightarrow Y$ smooth with $\dim X = \dim Y$ and Y connected. Recall that the mod 2 degree of f is defined in terms of the mod 2 intersection number as follows: $\deg_2(f) = I_2(f, \iota)$, where $\iota : y \mapsto Y$ is the inclusion map of a point $y \in Y$.

i) Prove that $\deg_2(f)$ is independent of the point $y \in Y$.

ii) If Y is non-compact, prove that $\deg_2(f) = 0$.

iii) A map $f : X \longrightarrow Y$ is called *essential* when it is not homotopic to a constant map. Prove that if $\deg_2(f) = 1$, then f is essential.

iv) Give example of a smooth surjective map $f : S^2 \longrightarrow S^2$ with $\deg_2(f) = 0$.

v) Can there exist a smooth map $f : S^2 \longrightarrow T^2$ with $\deg_2(f) = 1$? [Hint: consider two embedded circles C_1, C_2 in T^2 intersecting transversally at a single point.] Can there exist a smooth map of $\deg_2(f) = 1$ in the opposite direction? In each case, give proofs.

Question 4. Let $f : S^1 \longrightarrow \mathbb{R}^2$ be an embedding and choose $p \in \mathbb{R}^2 \backslash f(S^1)$. Define $f_p : S^1 \longrightarrow S^1$ by $f_p(z) = \frac{f(z)-p}{|f(z)-p|}$. Then we define the mod 2 winding number of f about p to be the degree of f_p, i.e. $w_2(f, p) = \deg_2(f_p)$. Warm up by computing $w_2(f, p)$ for the standard embedding of S^1 in $\mathbb{R}^2$, and for any p, with justifications.

i) Let $R_p(v)$ be the ray starting at p with direction $v \in S^1$. Prove that $v \in S^1$ is a critical value of f_p if and only if $R_p(v)$ is somewhere tangent to $f(S^1)$.

ii) Show that $w_2(f, p)$ coincides with the number of points mod 2 in $R_p(v) \cap f(S^1)$, whenever v is a regular value of f_p.

iii) Show that there are points $p, q \in \mathbb{R}^2 \backslash f(S^1)$ such that $w_2(f, p) = 0$ and $w_2(f, q) = 1$. Show that this implies that $\mathbb{R}^2 \backslash f(S^1)$ has at least two components.

iv) Fix $a \in f(S^1)$. Show that it is possible to choose a coordinate chart (U, φ) containing a such that $\varphi(U)$ contains $(-2, 2) \times (-2, 2)$, $\varphi(a) = (0, 0)$, and $\varphi(U \cap f(S^1)) = \{(x, y) \ : \ y = 0\}$.

v) Prove that each point $p \in \mathbb{R}^2 \backslash f(S^1)$ may be connected by a continuous path to either $\varphi^{-1}(0, 1)$ or $\varphi^{-1}(0, -1)$. [Hint: recall the tubular neighbourhood theorem]. Conclude that $\mathbb{R}^2 \backslash f(S^1)$ has two connected components. This is the smooth Jordan theorem.

Does the previous result (that $\mathbb{R}^2 \backslash f(S^1)$ has two connected components) hold for any embedding of a compact n-manifold $M \longrightarrow \mathbb{R}^{n+1}$? How so, or why not?