18.965 Differential Geometry, MIT Fall term, 2007

Partial solutions, Exercise 4.

There were some questions concerning the fact that the cohomology class $[\mathrm{Tr}(R^k)] \in H^{2k}(M, \mathbb{R})$ is independent of connection. Defining $\nabla' = \nabla + A$ where $A \in \Omega^1(\mathrm{End}(E))$.

Computing the new curvature, we have

$$R(\nabla') = R(\nabla) + d^{\nabla}_{\mathrm{End}} A + A \wedge A. \tag{1}$$

Remark 1. *At this point it is worth observing the following: the graded algebra* $\Omega^\bullet(\mathrm{End}(E))$ *is a graded Lie algebra, via*

$$[\rho \otimes A, \tau \otimes B] = (\rho \otimes A) \wedge (\tau \otimes B) - (-1)^{\rho\tau} (\tau \otimes B) \wedge (\rho \otimes A).$$

Also we observe that the trace of a commutator is zero:

$$\mathrm{Tr}[\varphi, \psi] = 0 \quad \forall \varphi, \psi \in \Omega^\bullet(\mathrm{End}(E)).$$

We may therefore rewrite Equation (1)*:*

$$R(\nabla') = R(\nabla) + d^{\nabla}_{\mathrm{End}} A + \tfrac{1}{2}[A, A].$$

One often sees this expression for the curvature. The graded Lie algebra notation is useful since we have the fact that

$$d^{\nabla'}_{\mathrm{End}} = d^{\nabla}_{\mathrm{End}} + [A, -],$$

that is, ∇' *differs from* ∇ *on endomorphism-valued forms by a commutator with* A*.*

Now we proceed with our calculation: first for $k = 1$ (we use the shorthand notation d when we mean $d^{\nabla}_{\mathrm{End}}$).

$$\begin{aligned} \mathrm{Tr}(R') - \mathrm{Tr}(R) &= \mathrm{Tr}(R' - R) \\ &= \mathrm{Tr}(dA + \tfrac{1}{2}[A, A]) \\ &= d\mathrm{Tr}(A), \end{aligned}$$

where we used the fact that $\mathrm{Tr}([A, A]) = 0$.

Now for $k = 2$, we have (using $\mathrm{Tr}([R', R]) = 0$ in the first line)

$$\begin{aligned} \mathrm{Tr}((R')^2) - \mathrm{Tr}(R^2) &= \mathrm{Tr}((R' - R)(R' + R)) \\ &= \mathrm{Tr}((dA + A \wedge A)(2R + dA + A \wedge A)) \\ &= \mathrm{Tr}(2dA \wedge R + dA \wedge dA + A \wedge A \wedge 2R + 2dA \wedge A \wedge A + A^4) \\ &= \mathrm{Tr}(2dA \wedge R + dA \wedge dA + A \wedge A \wedge 2R + 2dA \wedge A \wedge A), \end{aligned}$$

where we use finally the fact that $\mathrm{Tr}(A^4) = \frac{1}{2}\mathrm{Tr}[A, A \wedge A \wedge A] = 0$.

There are three natural 3-forms in the picture: $A \wedge R$, $A \wedge dA$, and $A \wedge A \wedge A$. A linear combination of these serves as apotential for the above 4-form:

$$\begin{aligned} d\mathrm{Tr}(2A \wedge R + A \wedge dA + \tfrac{2}{3}A \wedge A \wedge A) \\ &= \mathrm{Tr}(2dA \wedge R + dA \wedge dA - A \wedge d^2A + 2dA \wedge A \wedge A) \\ &= \mathrm{Tr}(2dA \wedge R + dA \wedge dA - A \wedge [R, A] + 2dA \wedge A \wedge A) \\ &= \mathrm{Tr}(2dA \wedge R + dA \wedge dA + 2A \wedge A \wedge R + 2dA \wedge A \wedge A), \end{aligned}$$

yielding precisely the required 4-form. Note that we used the fact that the curvature of $d^{\nabla}_{\mathrm{End}}$ is simply the commutator with the curvature of d^{∇}, which was a question in a previous section of the problem.

In general the form satisfying $d\omega_k = \mathrm{Tr}(R')^k - \mathrm{Tr}(R)^k$ can be conveniently expressed as follows:

$$\omega_k = k \int_0^1 \mathrm{Tr}(A \wedge (R + tdA + t^2 A \wedge A)^{k-1})dt.$$

This can be obtained by studying the family of connections $\nabla_t = \nabla + tA$, for t going from 0 to 1. You look at how the curvature changes infinitesimally

$$\frac{d}{dt} R(\nabla_t),$$

show it is exact, and then integrate it from 0 to 1.