18.965 Differential Geometry, MIT Fall term, 2007

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Exercise 1 (Review of Hilbert spaces)**.** The typical example of an infinite-dimensional Hilbert space is the space of square-integrable functions modulo a.e. zero functions, L^2, or the square-summable sequences, ℓ^2. We also make use of the Sobolev spaces H^k which consist of measurable functions whose first k (weak) derivatives exist and are in L^2, modulo a.e. zero functions. $H^0 = L^2$ by definition. Note that by changing the power from 2 to p we can define L^p and $H^{k,p}$ spaces but these are not Hilbert spaces, only Banach spaces.

Let us work with Hilbert spaces over $\mathbb{R}$, i.e. $\mathcal{H}$ is a real vector space equipped with a nondegenerate, real, positive-definite inner product $\langle \cdot, \cdot \rangle$ with associated norm $||v|| = \langle v, v \rangle^{1/2}$, and which is such that $\mathcal{H}$ is *complete*, i.e. Cauchy sequences converge (of course the norm induces on $\mathcal{H}$ a topology and we are using it.)

Begin by proving the Cauchy-Schwarz inequality for Hilbert spaces:

$$|\langle v, w \rangle|^2 \leq \langle v, v \rangle \langle w, w \rangle,$$

and conclude from it the triangle inequality for the norm. Note that although L^p spaces are not Hilbert spaces, they still have a generalization of the Cauchy-Schwarz inequality, called the Hölder inequality.

i) **(Vector subspaces of Hilbert spaces)**

- Give examples of subspaces of ℓ^2 which a) have infinite dimension and codimension, b) which have finite dimension, and c) which have finite codimension.
- Give an example of a proper subspace of ℓ^2 which is closed.
- Give an example of a proper subspace of ℓ^2 which is not closed.
- Give an example of a proper subspace of ℓ^2 which is dense.
- if $W \subset \mathcal{H}$ is a subspace, show $\overline{W}$ is a subspace and show

$$W^\perp = (\overline{W})^\perp.$$

  Finally show that $W \cap W^\perp = \{0\}$.
- Show that if $W \subset \mathcal{H}$ is a closed subspace, then W and $\mathcal{H}/W$ naturally inherit a Hilbert space structure.
- Is it possible that $\overline{W}/W$ be nonzero but finite-dimensional?

ii) **The unit sphere**. Let $S(\mathcal{H}) \subset \mathcal{H}$ be the unit sphere in $\mathcal{H}$.

- Show that $S(\mathcal{H})$ is closed. Show it is compact iff $\dim \mathcal{H} < \infty$.
- Show that a linear map of Hilbert spaces $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is continuous if and only if $F(S(\mathcal{H}_1))$ is bounded. (This is why such maps are sometimes called "bounded operators") Show this is equivalent to the inequality

$$||Fv||_{\mathcal{H}_2} \leq C||v||_{\mathcal{H}_1} \quad \forall v \in \mathcal{H}_1, \tag{1}$$

  for some constant C (independent of v).
- Suppose that $W \subset \mathcal{H}_1$ is a dense linear subspace and $F : W \longrightarrow \mathcal{H}_2$ is a continuous linear map. Show that F has a unique extension to a continuous linear map $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$.

- A linear map of Hilbert spaces $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is called *compact* when $F(S(\mathcal{H}_1))$ is compact in $\mathcal{H}_2$ (i.e. any bounded sequence has a subsequence mapped by F to a convergent sequence). Give an example of a compact operator $\ell^2 \longrightarrow \ell^2$ which is also an injection. Can it be surjective? Can its image be dense? Give proof/examples.

iii) **The continuous dual** The operator norm of a continuous linear map $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is defined as

$$||F|| := \sup_{v \in S(\mathcal{H}_1)} ||Fv||_{\mathcal{H}_2}.$$

Show that the composition of continuous linear operators is a continuous operation in the operator norm, i.e. for A, B continous linear operators, show

$$||A \circ B|| \leq ||A||||B||.$$

Let $\mathcal{H}'$ denote the continuous dual of $\mathcal{H}$, i.e. the space of continuous linear maps $L : \mathcal{H} \longrightarrow \mathbb{R}$, equipped with operator norm, viewing $\mathbb{R}$ as a Hilbert space.

• Show that the "dualization map" $v \mapsto v^* = \langle v, \cdot \rangle$ is an injective, norm-preserving continuous linear map $\mathcal{H} \longrightarrow \mathcal{H}'$.

The Riesz representation theorem states that the dualization map is an isomorphism of Hilbert spaces.

• Show that if $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a continous linear operator, then $F^* : \mathcal{H}_2' \longrightarrow \mathcal{H}_1'$ defined by

$$F^* \mu = \mu \circ F$$

is a continuous linear map. If F is injective, under what conditions is F^* surjective? Show that if F is injective and $\mathrm{Im} F$ is dense, then F^* is injective.

iv) **Weak convergence** A sequence $(x_n)_{n \in \mathbb{N}}$ is said to be *weakly convergent* to $x \in \mathcal{H}$ (we write $x_n \rightharpoonup x$) if

$$\lim_{n \to \infty} f(x_n - x) = 0 \quad \forall f \in \mathcal{H}'.$$

• Show that a convergent sequence in $\mathcal{H}$ is automatically weakly convergent. For this reason, sometimes usual convergence in $\mathcal{H}$ is called *strong convergence*.

• Give an example of a weakly convergent sequence in ℓ^2 which is not strongly convergent.

• Let $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a continuous linear map of Hilbert spaces. Show that if $x_n \rightharpoonup x$ then $Fx_n \rightharpoonup Fx$. If F is compact, show Fx_n converges strongly to Fx.

• Prove that any bounded sequence $(x_n)_{n \in \mathbb{N}}$ in a Hilbert space $\mathcal{H}$ has a weakly convergent subsequence. [Hint: Show that it suffices to show convergence $(x_{n_k}, y) \longrightarrow (x, y)$ for y in the closure $\overline{S}$ of the span S of (x_n).]

• If $x_n \rightharpoonup x$, show

$$||x|| \leq \liminf_{n \to \infty} ||x_n||.$$

Also show that $x_n \to x$ iff both $x_n \rightharpoonup x$ and

$$||x|| = \lim_{n \to \infty} ||x_n||.$$

Exercise 2 (Sobolev spaces and the weak derivative)**.** Let $E \longrightarrow M$ be a smooth real vector bundle equipped with a positive-definite metric and metric connection ∇, and equip M with a Riemannian metric, so that the Sobolev inner products $\langle \cdot, \cdot \rangle_{H^k}$ and norms $||s||_{H^k} := \langle s, s\rangle_{H^k}^{1/2}$ are well-defined on the vector space $C_c^\infty(M, E)$ of smooth sections with compact support. Recall that

$$||s||_{H^k}^2 := ||s||_{L^2}^2 + ||\nabla s||_{L^2}^2 + \cdots + ||\nabla^k s||_{L^2}^2,$$

where $\nabla^k : C^\infty(E) \longrightarrow C^\infty(\otimes^k T^* \otimes E)$ is the composition of ∇ with the connections $\nabla^{LC} \otimes 1 + 1 \otimes \nabla : C^\infty(\otimes^i T^* \otimes E) \longrightarrow C^\infty(\otimes^{i+1} T^* \otimes E)$ for $i = 1, \ldots, k-1$ (∇^{LC} is the Levi-Civita connection). The Sobolev Hilbert space $H^k(M, E)$ is then defined as the completion of $C_c^\infty(M, E)$ with respect to the norm $|| \cdot ||_{H^k}$.

i) **Sobolev dual spaces** Define $H^{-k}(M, E)$ as the continuous dual of $H^k(M, E)$, and always identify $H^0(M, E) = L^2(M, E)$ with its dual using the dualization map (we don't do this for the H^k, $k \neq 0$ for reasons which will become clear below).

- If $s \in C_c^\infty(M, E)$, show that for $0 \leq k \leq l$ there is a constant c (independent of s) with

$$||s||_{H^k} \leq c||s||_{H^l};$$

conclude that the identity map on $C_c^\infty(M, E)$ extends to a continuous injective linear map $\iota : H^l(M, E) \longrightarrow H^k(M, E)$ with dense image.

- For $0 \leq k \leq l$, show that the continuous dual of the above injection defines a map

$$\iota^* : H^{-k}(M, E) \longrightarrow H^{-l}(M, E)$$

which is also an injective continuous map with dense image. As a result we get, for all $s < t$, continuous injections with dense image

$$H^t(M, E) \hookrightarrow H^s(M, E).$$

- For $k > 0$ determine the restriction of $\iota^*\iota$ to $C_0^\infty(M, E)$, where $\iota : H^k(M, E) \longrightarrow H^0(M, E)$ is the injection above and $\iota^* : H^0(M, E) \longrightarrow H^{-k}(M, E)$. Does this coincide the dualization map $H^k \longrightarrow H^{-k}$? Why?

ii) **Sobolev embedding** The vector space of continuous sections $C^0(M, E)$ may be equipped with a norm, called the "sup norm", also called its "uniform C^0 norm", or "L^∞ norm":

$$||s||_\infty = \sup_{x \in M} |s(x)|,$$

where $|s(x)| = h(s(x), s(x))^{1/2}$ is the pointwise norm induced from the metric h chosen on E. Convergence in this norm is precisely what is called "uniform convergence" and since the uniform limit of continuous functions is continuous (why? think about it) we know that the space of continuous sections with finite C^0 norm actually forms a Banach space (but not a Hilbert space).

The vector space of k-times differentiable sections $C^k(M, E)$ may similarly be equipped with the uniform C^k norm

$$||s||_{C^k}^2 = ||s||_\infty^2 + ||\nabla s||_\infty^2 + \cdots + ||\nabla^k s||_\infty^2,$$

and the C^k sections with finite C^k norm also form a Banach space.

The C^k norms can be used to define a topology on the vector space $C^\infty(M, E)$, defining that (s_i) converges to $s \in C^\infty(M, E)$ if and only if, for any compact set $K \subset M$, $||s_i - s||_{C^k(K)} \to 0$ for all

k. This is called the topology of "compact convergence" or more pompously "uniform convergence on compacta" for the first k derivatives. This topology does not come from a norm, but does endow $C^\infty(M, E)$ with the structure of a *Fréchet space*.

• Show that the inclusions $C_c^\infty(M, E) \subset C_c^l(M, E) \subset C_c^k(M, E)$ are continuous for $l > k \geq 0$, with respect to the Fréchet, C^k and C^l norms respectively, and that there is a continuous inclusion $C_c^k(M, E) \subset H^k(M, E)$, but that $C_c^k(M, E)$ is not included in $H^{k+1}(M, E)$.

The continuous dual space of $C_c^\infty(M, E)$ with respect to the Fréchet topology is called the vector space of *distributions*, denoted $\mathcal{D}'$ because $C_c^\infty(M, E)$ is sometimes denoted $\mathcal{D}$ (and sometimes called the "test functions"). The vector space $\mathcal{D}'$ is equipped with the so-called "weak-star" topology, which defines (δ_i) to converge to 0 iff

$$\lim_{i \to \infty} \delta_i(s) = 0 \quad \forall s \in C_0^\infty(M, E).$$

• Let φ be any section of E which is integrable over any compact subset of M (i.e. it is "locally integrable"). Then show that

$$s \mapsto \int_M \langle \varphi, s \rangle v_g$$

defines a distribution.

• Not all distributions have this form: Let $\mu \in E_x^*$ for a fixed point $x \in M$, and define

$$\delta_\mu(s) = \mu(s_x)$$

for $s \in C_0^\infty(M, E)$. Show that δ_μ is a distribution, it is called the Dirac delta distribution at μ.

• Show that H^s is continuously included in $\mathcal{D}'$ for all $s \in \mathbb{Z}$. Is $C_0^\infty(M, E)$ dense in $\mathcal{D}'$?

The Sobolev embedding theorem states that if $s - k > \frac{n}{2}$ where $n = \dim M$, then there is a constant C_s such that

$$||u||_{C^k} \leq C_s ||u||_{H^s} \quad \forall u \in C_0^\infty(M, E).$$

• Conclude that if $s - k > \frac{n}{2}$ there is a continuous embedding $H^s(M, E) \hookrightarrow C^k(M, E)$.

The Rellich-Kondratchov Lemma, based on the Ascoli-Arzela theorem, states that if M is *compact*, then the continuous inclusion $H^{k+1}(M, E) \hookrightarrow H^k(M, E)$ is actually *compact*, for all k, and that furthermore, when $s - k > n/2$, the embedding $H^s(M, E) \hookrightarrow C^k(M, E)$ is compact.

• Draw a diagram which encapsulates all the continuous inclusions between the spaces C^k, C^∞, H^k, and $\mathcal{D}'$.

• Give an elementary proof (using basic calculus) of the fact that $H^1 \hookrightarrow C^0$ is continuous for $M = \mathbb{R}$ and E the trivial real line bundle. Show that if $M = [0, 1]$ then $H^1(M, \mathbb{R}) \hookrightarrow C^0(M, \mathbb{R})$ is compact.

• Is the unit sphere in $H^1(M, E)$ closed in $H^0(M, E)$? Why?.

Exercise 3 (Weak derivatives and the weak Laplacian)**.** We use the notation from the previous exercise.

- Let $R \in C_c^\infty(E^* \otimes F)$ be a compactly supported bundle map. Show that R defines a continuous linear operator
$$R : H^k(M,E) \longrightarrow H^k(M,F) \ \ \forall k \in \mathbb{Z}.$$

- Let $\nabla : C^\infty(E) \longrightarrow C^\infty(T^* \otimes E)$ be the connection as before. Show that for every $k \geq 1$, ∇ defines a continuous linear operator
$$\nabla : H^k(M,E) \longrightarrow H^{k-1}(M, T^* \otimes E) \ \ \forall k \geq 1,$$
and extend ∇ to $k \leq 0$ via
$$(\nabla s)(\varphi) = s(\nabla^* \varphi), \tag{2}$$
where ∇^* is the formal adjoint, which may be written as
$$\nabla^* = - \star \circ w \circ \nabla \circ \star,$$
where $\star : \Omega^k(E) \longrightarrow \Omega^{n-k}(E)$ is the Hodge star, $\nabla : C^\infty(\wedge^{n-1}T^* \otimes E) \longrightarrow C^\infty(T^* \otimes \wedge^{n-1}T^* \otimes E)$ is the connection on E coupled to the Levi-Civita connection, and w is the contraction $T^* \otimes \wedge^{n-1}T^* \otimes E \longrightarrow \wedge^n T^* \otimes E$.

  With this expression for ∇^*, prove for M compact that (2) gives a well-defined extension of ∇ to a continuous operator
$$\nabla : H^k(M,E) \longrightarrow H^{k-1}(M, T^* \otimes E) \ \ \forall k \in \mathbb{Z}.$$

- In fact, Equation (2) may be used to extend ∇ to a continuous map of all distributions for M compact:
$$\nabla : \mathcal{D}'(M,E) \longrightarrow \mathcal{D}'(M,E);$$
show this is indeed the case. This is called the *weak* or distributional derivative. Compute the distributional derivative of the dirac delta distribution δ_μ defined earlier.

- Let $\nabla : C^\infty(\wedge^k T^*) \longrightarrow C^\infty(T^* \otimes \wedge^k T^*)$ be the extension of the Levi-Civita connection to differential forms, and let ∇^* be its formal adjoint. The Weitzenböck formula relates the Laplacian $\Delta = dd^* + d^*d$ to the connection Laplacian $\nabla^*\nabla$ via
$$\Delta = \nabla^*\nabla + R_k,$$
where $R_k : \wedge^k T^* \longrightarrow \wedge^k T^*$ is a bundle map constructed from the Riemann curvature tensor. R_k is self-adjoint in the sense $g(R_k v, w) = g(v, R_k w)$ at each point, where g is the induced Riemannian metric on forms. It is easily seen that $R_0 = R_n = 0$.

  Deduce for M compact that Δ extends to a continuous operator, for all p:
$$\Delta : H^k(M, \wedge^p T^*) \longrightarrow H^{k-2}(M, \wedge^p T^*) \ \ \forall k \in \mathbb{Z}.$$

- Let M be compact. Show that if, for all $0 < k < n$ we have that R_k is everywhere positive, i.e. $g(R_k v, v) > 0 \ \ \forall v$ at all points, then all Betti numbers $b_1, \ldots, b_{n-1}$ of M must vanish, i.e. M must be a homology sphere. This is known as Bochner-Gallot-Meyer theorem. Show it also holds if R is everywhere non-negative but positive at (at least) a point $x \in M$. This demonstrates the overwhelming topological restriction implied by the positivity (in a certain sense) of the Riemann curvature.