18.965 Differential Geometry, MIT Fall term, 2007

Differentialgeometrieexerzitien IV, due: Nov. 19, during class.

I encourage you to work together on the ideas but the solutions must be *individual*. Also, please be *concise* – if your proof is convoluted and takes up several pages, you are probably missing the point. Late assignments will not be accepted without *prior* arrangement.

Exercise 1 (Connections on real vector bundles)**.** We use $C^\infty(V)$ as a short hand for $C^\infty(M, V)$. A connection on a real vector bundle $V \longrightarrow M$ is a $\mathbb{R}$-linear map

$$\nabla : C^\infty(V) \longrightarrow C^\infty(T^* \otimes V),$$

such that $\nabla(fs) = f\nabla s + df \otimes s$ for $f \in C^\infty(M)$ and $s \in C^\infty(V)$. When composed with the contraction $X : T^* \otimes V \longrightarrow V$ defined by a vector field $X \in C^\infty(T)$, we write $\nabla_X : C^\infty(V) \longrightarrow C^\infty(V)$ and call $\nabla_X s$ the *covariant derivative* of s in the X direction. If $\nabla s = 0$, i.e. $\nabla_X s = 0$ for all X, we say that s is *covariant constant* or *parallel*.

Given a basis of sections $\{e_1, \ldots, e_r\}$ for the bundle V over the open set $U \subset M$ (this is called a *local trivialization* of V over U), we may write any section as $s = s^i e_i$ and $\nabla s = (ds^i + A_i^j s^i) \otimes e_i$, for $A_i^j \in \Omega^1(U)$ given by $\nabla e_i = \sum_j A_i^j \otimes e_j$. Therefore, given the isomorphism $V|_U \longrightarrow U \times \mathbb{R}^r$ we may write $\nabla = d + A$, for $A \in \Omega^1(U, \mathrm{End}(\mathbb{R}^r))$.

Finally, let $\mathcal{A}(V)$ denote the set of all connections on V.

i) Let $\nabla, \nabla' \in \mathcal{A}(V)$. Show that $\nabla' - \nabla$ is a tensor $A \in C^\infty(T^* \otimes \mathrm{End}(V))$. Show that $\nabla + A$ is a connection for all $A \in C^\infty(T^* \otimes \mathrm{End}(V))$. This shows that $\mathcal{A}(V)$ is an affine space modeled on the vector space $C^\infty(T^* \otimes \mathrm{End}(V))$. Conclude that $t\nabla + (1-t)\nabla' \in \mathcal{A}(V)$ for all $t \in \mathbb{R}$.

ii) Use the above to prove that any vector bundle over a manifold admits a connection, i.e. $\mathcal{A}(V)$ is nonempty for all V. Hint: Use a partition of unity.

iii) Let $\{U_i\}$ be an open cover for M and let the vector bundle V be locally trivialized over these open sets so that V is isomorphic to $U_i \times \mathbb{R}^r$ over each U_i, glued together by smooth transition functions

$$g_{ij} : U_i \cap U_j \longrightarrow GL_r(\mathbb{R}).$$

If the connection ∇ has expressions $d + A_i$ in each U_i, determine the constraints on the local connection 1-forms $A_i \in \Omega^1(U_i, \mathrm{End}(\mathbb{R}^r))$ so that they glue to a global connection on V.

iv) Let $F : M \longrightarrow N$ be smooth, and $V \longrightarrow N$ be a vector bundle with connection ∇. Show that the equation

$$(F^*\nabla)_X(F^*s) = F^*(\nabla_{F_*X}s),$$

for $X \in C^\infty(M, T)$ and $s \in C^\infty(V)$, uniquely defines a connection $F^*\nabla$ on the pullback bundle $F^*V \longrightarrow M$.

v) The curvature of a connection $\nabla \in \mathcal{A}(V)$ is the operator, for $X, Y \in C^\infty(T)$,

$$R^\nabla_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Show that R^∇ is a tensor in $\Omega^2(\mathrm{End}(V)) = C^\infty(\wedge^2 T^* \otimes \mathrm{End}(V))$. Don't forget to show that R^∇ is tensorial in X, Y and also in its action on sections $s \in C^\infty(V)$.

vi) Let $\Omega^k(V) = C^\infty(M, \wedge^k T^* \otimes V)$, the "forms with values in V", so that $\Omega^\bullet(V) = \oplus_k \Omega^k(V)$ is a graded vector space which is also a graded module over the graded algebra $\Omega^\bullet(M)$, via the action, for $\rho \in \Omega^k(M)$, $\tau \in \Omega^l(M)$ and $s \in C^\infty(V)$:

$$\rho \cdot (\tau \otimes s) = (\rho \wedge \tau) \otimes s.$$

Show that the choice of a connection $\nabla \in \mathcal{A}(V)$ induces a degree +1 operator d^∇ on $\Omega^\bullet(V)$ via

$$d^\nabla(\tau \otimes s) = d\tau \otimes s + (-1)^{|\tau|}\tau \overset{\otimes}{\wedge} \nabla s,$$

where $\overset{\otimes}{\wedge}$ denotes the map $\wedge^k T^* \otimes T^* \otimes V \longrightarrow \wedge^{k+1} T^* \otimes V$ given by $\alpha \otimes \mu \otimes s \mapsto (\alpha \wedge \mu) \otimes s$. Show that d^∇ is a graded derivation of the structure of $\Omega^\bullet(V)$ as a $\Omega^\bullet(M)$-module.

vii) Prove that

$$(d^\nabla)^2 = R^\nabla \cdot, \tag{1}$$

where $R^\nabla \in \Omega^2(\mathrm{End}(V))$ acts on $\Omega^\bullet(V)$ via wedge product and the endomorphism action on V, i.e. $(\rho \otimes A) \cdot (\tau \otimes s) = (\rho \wedge \tau) \otimes (As)$.

When $R^\nabla = 0$ we say that (V, ∇) is a *flat bundle*, and by Equation (1) we see that d^∇ defines a differential complex. Its cohomology is called the *de Rham cohomology with coefficients in the flat bundle V*.

viii) Given a connection $\nabla \in \mathcal{A}(V)$, there is a natural connection on the dual bundle

$$\nabla^* : C^\infty(V^*) \longrightarrow C^\infty(T^* \otimes V^*),$$

defined, for $\xi \in C^\infty(V^*), X \in C^\infty(T)$ by

$$(\nabla^*_X \xi)(s) = X(\xi(s)) - \xi(\nabla_X s),$$

where we have paired it with the section $s \in C^\infty(V)$. Show this defines a connection. How does its curvature differ from that of ∇? If $\{e_1, \ldots, e_r\}$ is a local basis of sections for V with connection 1-forms $A_i^j \in \Omega^1(U)$, what are the connection 1-forms for the dual basis $\{e^1, \ldots, e^r\}$ for V^*?

ix) Given connections ∇_1, ∇_2 on vector bundles V_1, V_2 respectively, show that $\nabla = \nabla_1 \otimes 1 + 1 \otimes \nabla_2$ defines a connection on $V_1 \otimes V_2$, being careful to interpret ∇ correctly. Conclude that a connection $\nabla \in \mathcal{A}(V)$ induces a connection ∇_{End} on the bundle $\mathrm{End}(V)$. Show the induced operator d^∇_{End} on $\Omega^\bullet(\mathrm{End}(V))$ is a graded derivation of the graded algebra structure

$$(\rho \otimes A) \wedge (\tau \otimes B) = (\rho \wedge \tau) \otimes (AB)$$

on $\Omega^\bullet(\mathrm{End}(V))$. How is the curvature of ∇_{End} related to the curvature of ∇?

x) Let $\nabla \in \mathcal{A}(V)$. Prove that $d^\nabla_{\mathrm{End}}(R^\nabla) = 0$. This is the "Bianchi identity".

xi) Let $\nabla' = \nabla + A$, for $A \in \Omega^1(\mathrm{End}(V))$, and show that

$$R(\nabla') = R(\nabla) + d^\nabla_{\mathrm{End}} A + A \wedge A.$$

xii) Let $\mathrm{Tr} : \Omega^\bullet(\mathrm{End}(V)) \longrightarrow \Omega^\bullet(M)$ be defined by

$$\mathrm{Tr}(\rho \otimes A) \mapsto \mathrm{Tr}(A)\rho.$$

Verify that Tr is well-defined and prove that $d \circ \mathrm{Tr} = \mathrm{Tr} \circ d^\nabla_{\mathrm{End}}$.

xiii) Deduce from $f), g), h)$ that $\mathrm{Tr}((R^\nabla)^k)$ defines a sequence of closed forms on the manifold M. (These are called "Chern-Weil" forms and their cohomology classes are multiples of so-called "characteristic classes" of the bundle V.)

xiv) Show that $[\mathrm{Tr}((R^\nabla)^k)] \in H^{2k}_{dR}(M)$ are independent of the choice of connection ∇. Hint: replace ∇ by $\nabla' = \nabla + A$ and use the formula for $R^{\nabla'}$ above.

Exercise 2 (Rank 2 bundles on the 2-sphere)**.** Express the 2-sphere as the gluing of $U_0 = \mathbb{C}$ to $U_1 = \mathbb{C}$ along $U_0 \cap U_1 = \mathbb{C} - \{0\}$ via the map $z_0 \mapsto 1/z_0$ (here z_0 is the coordinate on $U_0 = \mathbb{C}$). Let z_1 be the standard coordinate on U_1. Show that

$$\theta_0 = \frac{1}{1 + z_0\bar{z}_0},$$
$$\theta_1 = \frac{1}{1 + z_1\bar{z}_1}$$

defines a partition of unity.

Let $\mathcal{O}(k),\quad k \in \mathbb{Z}$ denote the real 2-plane bundle (viewed as a complex line bundle) which is the gluing of the trivial bundles $U_0 \times \mathbb{C}$, $U_1 \times \mathbb{C}$ via the transition function $g_{01} : (U_0 \cap U_1) \times \mathbb{C} \longrightarrow (U_0 \cap U_1) \times \mathbb{C}$ given by

$$g_{01} : (z_0, v) \mapsto (z_0, z_0^k v),$$

where we use the coordinate z_0 on U_0 to describe the map.

Use the partition of unity to construct a complex connection on $\mathcal{O}(k)$, then compute its curvature $R^\nabla \in \Omega^2(M, \mathbb{C})$, and then compute

$$\tfrac{i}{2\pi} \int_{S^2} R^\nabla$$

Conclude that the bundles $\mathcal{O}(k)$ are all non-isomorphic. The above integral is the *first Chern class* of the bundle $\mathcal{O}(k)$, the most important example of a characteristic class.

Exercise 3 (The Levi-Civita connection)**.** We now focus on the tangent bundle, taking $V = T$ in the first problem. If we choose a coordinate system $\{x^1, \ldots, x^n\}$ for $U \subset M$, we abbreviate $\partial/\partial x^i$ as ∂_i and obtain a local basis $\{\partial_1, \ldots, \partial_n\}$ of sections of T, and so we may write $\nabla(\partial_i) = \theta_i^j \otimes \partial_j$, for 1-forms θ_i^j, and even more explicitly we may write

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k,$$

for $\Gamma^k_{ij} = \theta^k_j(\partial_i)$. The functions $\Gamma^k_{ij} \in \Omega^0(U)$ are called the “Christoffel symbols”.

i) Let $\nabla \in \mathcal{A}(T)$. Show that

$$T_{X,Y} = \nabla_X Y - \nabla_Y X - [X,Y]$$

defines a tensor $T \in \Omega^2(T) = C^\infty(M, \wedge^2 T^* \otimes T)$. This is called the "torsion" of ∇. Note that the use of the Lie bracket $[\cdot,\cdot]$ is specific to the tangent bundle, so torsion does not make sense for an arbitrary vector bundle. Express the torsion in terms of the Christoffel symbols.

ii) Let $g \in C^\infty(T^* \otimes T^*)$ be a (pseudo-)Riemannian metric on M. Then $\nabla \in \mathcal{A}(T)$ is called a *metric* connection when $\nabla g = 0$, i.e. g is parallel with respect to the induced connection on $T^* \otimes T^*$. Show that this is equivalent to the requirement

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \quad \forall X, Y, Z \in C^\infty(T). \tag{2}$$

iii) Suppose ∇ is a metric connection on T with vanishing torsion. Show that ∇ must be unique by considering a general connection $\nabla' = \nabla + A$ and placing conditions on $A \in C^\infty(T^* \otimes \mathrm{End}(T))$. We call this unique connection the *Levi-Civita* connection.

iv) By using Equation (2) and its cyclic permutations in X, Y, Z, and using the vanishing of the torsion, write an expression defining the Levi-Civita connection explicitly, showing that it does, in fact, exist. Do this without the use of coordinates.

v) Find the Christoffel symbols of the Levi-Civita connection in a coordinate system, in terms of the metric $g = g_{ij} dx^i \otimes dx^j$.

vi) Compute the curvature tensor of the Levi-Civita connection, $R = R_{ijk}{}^l (dx^i \wedge dx^j) \otimes (dx^k \otimes \partial_l)$ in $\Omega^2(\mathrm{End}(T))$, in terms of the Christoffel symbols. This is called the *Riemann tensor*.

Exercise 4 (Geodesics)**.** Let (M, g) be a (Pseudo)-Riemannian manifold, and let ∇ be the Levi-Civita connection as in the previous problem.

In class we defined geodesics as curves $\gamma : I \longrightarrow M$, $I \subset \mathbb{R}$ an interval, which are projections of integral curves $\tilde{\gamma} : I \longrightarrow T^*M$ via $\pi : T^*M \longrightarrow M$, for the Hamiltonian vector field X_H associated to the function

$$H(x,p) = g^{-1}(p,p)$$

on T^*M. Here g^{-1} is a metric on T^*M obtained from the metric g on TM via the isomorphism $g : T \longrightarrow T^*$ given by $X \mapsto g(X, \cdot)$. In coordinates $g^{-1} = g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$, for g^{ij} defined by $g^{ij} g_{jk} = \delta^i_k$.

Choosing coordinates $(x^1, \ldots, x^n)$ for $U \subset M$, we obtain canonical coordinates $(x, p) = (x^1, \ldots, x^n, p_1, \ldots p_n)$ for points $p_i dx^i$ in T^*U. In this coordinate chart we may write the ODE governing the integral curve $\gamma(t) = (x(t), p(t))$ as

$$\dot{x}^i = 2g^{ij}p_j$$
$$\dot{p}^k = -p_i p_j \tfrac{\partial g^{ij}}{\partial x^k},$$

known as the *Hamilton-Jacobi* equations.

i) If g is Riemannian, consider the unit sphere bundle of the cotangent bundle
$$S(T^*M) = \{(x,p) \in T^*M \ : \ g(p,p) = 1\}.$$
Show that X_H is tangent to $S(T^*M)$, and prove from this that any compact Riemannian manifold is geodesically complete (i.e. every geodesic may be extended to a geodesic with domain all of $\mathbb{R}$).

ii) By substitution of the Hamilton-Jacobi equations, obtain a second order ODE of the form
$$\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0. \tag{3}$$
Give Γ^k_{ij} in terms of g_{ij} and prove that they coincide with the Christoffel symbols from the previous exercise.

iii) Prove that Equation (3) is equivalent to the equation, for the curve $\gamma : I \longrightarrow M$, that
$$(\gamma^*\nabla)(\gamma_* \tfrac{\partial}{\partial t}) = 0. \tag{4}$$
Note: $\gamma_* \frac{\partial}{\partial t}$ is usually denoted $\dot{\gamma}$ and is the derivative of the curve, viewed as a section of the pullback of TM via γ; $\gamma^*\nabla$ is the pullback Levi-Civita connection on γ^*TM. Equation (4) is called the *geodesic equation* and is the statement that the curve has parallel velocity, i.e. *it is the path of an observer experiencing no acceleration.*

iv) Let f be a smooth function on the interval $[a,b] \subset \mathbb{R}$ and suppose
$$\int_a^b f(x)h(x)dx = 0$$
for all smooth functions h such that $h(a) = h(b) = 0$. Show that f must be identically zero.

v) Show that $\gamma : [a, b] \longrightarrow M$ is a geodesic (in any of the equivalent senses above) if and only if it is an extremum of the energy functional, which is a real-valued function on the space of paths $\gamma : [a, b] \longrightarrow M$ with fixed endpoints $\gamma(a), \gamma(b) \in M$, given by

$$\mathcal{E}(\gamma) = \int_a^b g(\dot{\gamma}, \dot{\gamma}) dt.$$

For simplicity you may assume that the image of γ lies entirely in one coordinate chart $U \subset M$ and that the definition of *extremum* is that the derivative with respect to $\epsilon \in \mathbb{R}$ of the usual function $\mathbb{R} \longrightarrow \mathbb{R}$

$$e(\epsilon) = \mathcal{E}(\gamma + \epsilon\sigma) \tag{5}$$

at $\epsilon = 0$, is zero, for all smooth curves $\sigma : [a, b] \longrightarrow \mathbb{R}^n$ with $\sigma(a) = \sigma(b) = 0$. Note that we are using the additive structure of $\mathbb{R}^n$ to simplify our calculation. Explain why $e(\epsilon)$ is in fact smooth in a neighbourhood of $\epsilon = 0$.

vi) Let $j : S^1 \times S^1 \longrightarrow \mathbb{R}^3$ be the embedding of the torus given by

$$j(\theta_1, \theta_2) = ((1 + \tfrac{r}{R}\cos\theta_2)R\cos\theta_1, (1 + \tfrac{r}{R}\cos\theta_2)R\sin\theta_1, r\sin\theta_2)$$

where $0 < r < R$ are real constants.

Using the pullback Euclidean metric $j^* g_{\mathbb{R}^3}$, find a closed geodesic curve $\gamma : S^1 \longrightarrow S^1 \times S^1$ in the homotopy class of the sum of the two generating circles of $S^1 \times S^1$, and compute its length.

For those interested, you may take a look at Klingenberg's "Closed Geodesics on Riemannian manifolds" for a treatment of several amazing results including the existence of infinitely many prime (non-repeating) closed geodesics on any compact Riemannian manifold with finite fundamental group.