

## 18.965 Differential Geometry, MIT Fall term, 2007

### Differentialgeometrieeexercitien IV, due: Nov. 19, during class.

I encourage you to work together on the ideas but the solutions must be *individual*. Also, please be *concise* – if your proof is convoluted and takes up several pages, you are probably missing the point. Late assignments will not be accepted without *prior* arrangement.

**Exercise 1** (Connections on real vector bundles). We use  $C^\infty(V)$  as a short hand for  $C^\infty(M, V)$ . A connection on a real vector bundle  $V \rightarrow M$  is a  $\mathbb{R}$ -linear map

$$\nabla : C^\infty(V) \rightarrow C^\infty(T^* \otimes V),$$

such that  $\nabla(fs) = f\nabla s + df \otimes s$  for  $f \in C^\infty(M)$  and  $s \in C^\infty(V)$ . When composed with the contraction  $X : T^* \otimes V \rightarrow V$  defined by a vector field  $X \in C^\infty(T)$ , we write  $\nabla_X : C^\infty(V) \rightarrow C^\infty(V)$  and call  $\nabla_X s$  the *covariant derivative* of  $s$  in the  $X$  direction. If  $\nabla s = 0$ , i.e.  $\nabla_X s = 0$  for all  $X$ , we say that  $s$  is *covariant constant* or *parallel*.

Given a basis of sections  $\{e_1, \dots, e_r\}$  for the bundle  $V$  over the open set  $U \subset M$  (this is called a *local trivialization* of  $V$  over  $U$ ), we may write any section as  $s = s^i e_i$  and  $\nabla s = (ds^i + A_i^j s^i) \otimes e_j$ , for  $A_i^j \in \Omega^1(U)$  given by  $\nabla e_i = \sum_j A_i^j \otimes e_j$ . Therefore, given the isomorphism  $V|_U \rightarrow U \times \mathbb{R}^r$  we may write  $\nabla = d + A$ , for  $A \in \Omega^1(U, \text{End}(\mathbb{R}^r))$ .

Finally, let  $\mathcal{A}(V)$  denote the set of all connections on  $V$ .

- i) Let  $\nabla, \nabla' \in \mathcal{A}(V)$ . Show that  $\nabla' - \nabla$  is a tensor  $A \in C^\infty(T^* \otimes \text{End}(V))$ . Show that  $\nabla + A$  is a connection for all  $A \in C^\infty(T^* \otimes \text{End}(V))$ . This shows that  $\mathcal{A}(V)$  is an affine space modeled on the vector space  $C^\infty(T^* \otimes \text{End}(V))$ . Conclude that  $t\nabla + (1-t)\nabla' \in \mathcal{A}(V)$  for all  $t \in \mathbb{R}$ .
- ii) Use the above to prove that any vector bundle over a manifold admits a connection, i.e.  $\mathcal{A}(V)$  is nonempty for all  $V$ . Hint: Use a partition of unity.
- iii) Let  $\{U_i\}$  be an open cover for  $M$  and let the vector bundle  $V$  be locally trivialized over these open sets so that  $V$  is isomorphic to  $U_i \times \mathbb{R}^r$  over each  $U_i$ , glued together by smooth transition functions

$$g_{ij} : U_i \cap U_j \rightarrow GL_r(\mathbb{R}).$$

If the connection  $\nabla$  has expressions  $d + A_i$  in each  $U_i$ , determine the constraints on the local connection 1-forms  $A_i \in \Omega^1(U_i, \text{End}(\mathbb{R}^r))$  so that they glue to a global connection on  $V$ .

- iv) Let  $F : M \rightarrow N$  be smooth, and  $V \rightarrow N$  be a vector bundle with connection  $\nabla$ . Show that the equation

$$(F^*\nabla)_X(F^*s) = F^*(\nabla_{F_*X}s),$$

for  $X \in C^\infty(M, T)$  and  $s \in C^\infty(V)$ , uniquely defines a connection  $F^*\nabla$  on the pullback bundle  $F^*V \rightarrow M$ .

- v) The curvature of a connection  $\nabla \in \mathcal{A}(V)$  is the operator, for  $X, Y \in C^\infty(T)$ ,

$$R_{X,Y}^\nabla = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Show that  $R^\nabla$  is a tensor in  $\Omega^2(\text{End}(V)) = C^\infty(\wedge^2 T^* \otimes \text{End}(V))$ . Don't forget to show that  $R^\nabla$  is tensorial in  $X, Y$  and also in its action on sections  $s \in C^\infty(V)$ .

- vi) Let  $\Omega^k(V) = C^\infty(M, \wedge^k T^* \otimes V)$ , the “forms with values in  $V$ ”, so that  $\Omega^\bullet(V) = \bigoplus_k \Omega^k(V)$  is a graded vector space which is also a graded module over the graded algebra  $\Omega^\bullet(M)$ , via the action, for  $\rho \in \Omega^k(M)$ ,  $\tau \in \Omega^l(M)$  and  $s \in C^\infty(V)$ :

$$\rho \cdot (\tau \otimes s) = (\rho \wedge \tau) \otimes s.$$

Show that the choice of a connection  $\nabla \in \mathcal{A}(V)$  induces a degree +1 operator  $d^\nabla$  on  $\Omega^\bullet(V)$  via

$$d^\nabla(\tau \otimes s) = d\tau \otimes s + (-1)^{|\tau|} \tau \overset{\otimes}{\wedge} \nabla s,$$

where  $\overset{\otimes}{\wedge}$  denotes the map  $\wedge^k T^* \otimes T^* \otimes V \rightarrow \wedge^{k+1} T^* \otimes V$  given by  $\alpha \otimes \mu \otimes s \mapsto (\alpha \wedge \mu) \otimes s$ . Show that  $d^\nabla$  is a graded derivation of the structure of  $\Omega^\bullet(V)$  as a  $\Omega^\bullet(M)$ -module.

- vii) Prove that

$$(d^\nabla)^2 = R^\nabla \cdot, \tag{1}$$

where  $R^\nabla \in \Omega^2(\text{End}(V))$  acts on  $\Omega^\bullet(V)$  via wedge product and the endomorphism action on  $V$ , i.e.  $(\rho \otimes A) \cdot (\tau \otimes s) = (\rho \wedge \tau) \otimes (As)$ .

When  $R^\nabla = 0$  we say that  $(V, \nabla)$  is a *flat bundle*, and by Equation (1) we see that  $d^\nabla$  defines a differential complex. Its cohomology is called the *de Rham cohomology with coefficients in the flat bundle  $V$* .

- viii) Given a connection  $\nabla \in \mathcal{A}(V)$ , there is a natural connection on the dual bundle

$$\nabla^* : C^\infty(V^*) \longrightarrow C^\infty(T^* \otimes V^*),$$

defined, for  $\xi \in C^\infty(V^*)$ ,  $X \in C^\infty(T)$  by

$$(\nabla_X^* \xi)(s) = X(\xi(s)) - \xi(\nabla_X s),$$

where we have paired it with the section  $s \in C^\infty(V)$ . Show this defines a connection. How does its curvature differ from that of  $\nabla$ ? If  $\{e_1, \dots, e_r\}$  is a local basis of sections for  $V$  with connection 1-forms  $A_i^j \in \Omega^1(U)$ , what are the connection 1-forms for the dual basis  $\{e^1, \dots, e^r\}$  for  $V^*$ ?

- ix) Given connections  $\nabla_1, \nabla_2$  on vector bundles  $V_1, V_2$  respectively, show that  $\nabla = \nabla_1 \otimes 1 + 1 \otimes \nabla_2$  defines a connection on  $V_1 \otimes V_2$ , being careful to interpret  $\nabla$  correctly. Conclude that a connection  $\nabla \in \mathcal{A}(V)$  induces a connection  $\nabla_{\text{End}}$  on the bundle  $\text{End}(V)$ . Show the induced operator  $d_{\text{End}}^\nabla$  on  $\Omega^\bullet(\text{End}(V))$  is a graded derivation of the graded algebra structure

$$(\rho \otimes A) \wedge (\tau \otimes B) = (\rho \wedge \tau) \otimes (AB)$$

on  $\Omega^\bullet(\text{End}(V))$ . How is the curvature of  $\nabla_{\text{End}}$  related to the curvature of  $\nabla$ ?

- x) Let  $\nabla \in \mathcal{A}(V)$ . Prove that  $d_{\text{End}}^\nabla(R^\nabla) = 0$ . This is the ‘‘Bianchi identity’’.
- xi) Let  $\nabla' = \nabla + A$ , for  $A \in \Omega^1(\text{End}(V))$ , and show that

$$R(\nabla') = R(\nabla) + d_{\text{End}}^\nabla A + A \wedge A.$$

- xii) Let  $\text{Tr} : \Omega^\bullet(\text{End}(V)) \longrightarrow \Omega^\bullet(M)$  be defined by

$$\text{Tr}(\rho \otimes A) \mapsto \text{Tr}(A)\rho.$$

Verify that  $\text{Tr}$  is well-defined and prove that  $d \circ \text{Tr} = \text{Tr} \circ d_{\text{End}}^\nabla$ .

- xiii) Deduce from *f), g), h)* that  $\text{Tr}((R^\nabla)^k)$  defines a sequence of closed forms on the manifold  $M$ . (These are called ‘‘Chern-Weil’’ forms and their cohomology classes are multiples of so-called ‘‘characteristic classes’’ of the bundle  $V$ .)

- xiv) Show that  $[\text{Tr}((R^\nabla)^k)] \in H_{dR}^{2k}(M)$  are independent of the choice of connection  $\nabla$ . Hint: replace  $\nabla$  by  $\nabla' = \nabla + A$  and use the formula for  $R^{\nabla'}$  above.

**Exercise 2** (Rank 2 bundles on the 2-sphere). Express the 2-sphere as the gluing of  $U_0 = \mathbb{C}$  to  $U_1 = \mathbb{C}$  along  $U_0 \cap U_1 = \mathbb{C} - \{0\}$  via the map  $z_0 \mapsto 1/z_0$  (here  $z_0$  is the coordinate on  $U_0 = \mathbb{C}$ ). Let  $z_1$  be the standard coordinate on  $U_1$ . Show that

$$\theta_0 = \frac{1}{1 + z_0 \bar{z}_0},$$

$$\theta_1 = \frac{1}{1 + z_1 \bar{z}_1}$$

defines a partition of unity.

Let  $\mathcal{O}(k)$ ,  $k \in \mathbb{Z}$  denote the real 2-plane bundle (viewed as a complex line bundle) which is the gluing of the trivial bundles  $U_0 \times \mathbb{C}$ ,  $U_1 \times \mathbb{C}$  via the transition function  $g_{01} : (U_0 \cap U_1) \times \mathbb{C} \rightarrow (U_0 \cap U_1) \times \mathbb{C}$  given by

$$g_{01} : (z_0, v) \mapsto (z_0, z_0^k v),$$

where we use the coordinate  $z_0$  on  $U_0$  to describe the map.

Use the partition of unity to construct a complex connection on  $\mathcal{O}(k)$ , then compute its curvature  $R^\nabla \in \Omega^2(M, \mathbb{C})$ , and then compute

$$\frac{i}{2\pi} \int_{S^2} R^\nabla$$

Conclude that the bundles  $\mathcal{O}(k)$  are all non-isomorphic. The above integral is the *first Chern class* of the bundle  $\mathcal{O}(k)$ , the most important example of a characteristic class.

**Exercise 3** (The Levi-Civita connection). We now focus on the tangent bundle, taking  $V = T$  in the first problem. If we choose a coordinate system  $\{x^1, \dots, x^n\}$  for  $U \subset M$ , we abbreviate  $\partial/\partial x^i$  as  $\partial_i$  and obtain a local basis  $\{\partial_1, \dots, \partial_n\}$  of sections of  $T$ , and so we may write  $\nabla(\partial_i) = \theta_i^j \otimes \partial_j$ , for 1-forms  $\theta_i^j$ , and even more explicitly we may write

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k,$$

for  $\Gamma_{ij}^k = \theta_j^k(\partial_i)$ . The functions  $\Gamma_{ij}^k \in \Omega^0(U)$  are called the ‘‘Christoffel symbols’’.

i) Let  $\nabla \in \mathcal{A}(T)$ . Show that

$$T_{X,Y} = \nabla_X Y - \nabla_Y X - [X, Y]$$

defines a tensor  $T \in \Omega^2(T) = C^\infty(M, \wedge^2 T^* \otimes T)$ . This is called the “torsion” of  $\nabla$ . Note that the use of the Lie bracket  $[\cdot, \cdot]$  is specific to the tangent bundle, so torsion does not make sense for an arbitrary vector bundle. Express the torsion in terms of the Christoffel symbols.

ii) Let  $g \in C^\infty(T^* \otimes T^*)$  be a (pseudo-)Riemannian metric on  $M$ . Then  $\nabla \in \mathcal{A}(T)$  is called a *metric* connection when  $\nabla g = 0$ , i.e.  $g$  is parallel with respect to the induced connection on  $T^* \otimes T^*$ . Show that this is equivalent to the requirement

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \quad \forall X, Y, Z \in C^\infty(T). \quad (2)$$

iii) Suppose  $\nabla$  is a metric connection on  $T$  with vanishing torsion. Show that  $\nabla$  must be unique by considering a general connection  $\nabla' = \nabla + A$  and placing conditions on  $A \in C^\infty(T^* \otimes \text{End}(T))$ . We call this unique connection the *Levi-Civita* connection.

iv) By using Equation (2) and its cyclic permutations in  $X, Y, Z$ , and using the vanishing of the torsion, write an expression defining the Levi-Civita connection explicitly, showing that it does, in fact, exist. Do this without the use of coordinates.

v) Find the Christoffel symbols of the Levi-Civita connection in a coordinate system, in terms of the metric  $g = g_{ij} dx^i \otimes dx^j$ .

vi) Compute the curvature tensor of the Levi-Civita connection,  $R = R_{ijk}{}^l (dx^i \wedge dx^j) \otimes (dx^k \otimes \partial_l)$  in  $\Omega^2(\text{End}(T))$ , in terms of the Christoffel symbols. This is called the *Riemann tensor*.

**Exercise 4 (Geodesics).** Let  $(M, g)$  be a (Pseudo)-Riemannian manifold, and let  $\nabla$  be the Levi-Civita connection as in the previous problem.

In class we defined geodesics as curves  $\gamma : I \rightarrow M$ ,  $I \subset \mathbb{R}$  an interval, which are projections of integral curves  $\tilde{\gamma} : I \rightarrow T^*M$  via  $\pi : T^*M \rightarrow M$ , for the Hamiltonian vector field  $X_H$  associated to the function

$$H(x, p) = g^{-1}(p, p)$$

on  $T^*M$ . Here  $g^{-1}$  is a metric on  $T^*M$  obtained from the metric  $g$  on  $TM$  via the isomorphism  $g : T \rightarrow T^*$  given by  $X \mapsto g(X, \cdot)$ . In coordinates  $g^{-1} = g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$ , for  $g^{ij}$  defined by  $g^{ij} g_{jk} = \delta_k^i$ .

Choosing coordinates  $(x^1, \dots, x^n)$  for  $U \subset M$ , we obtain canonical coordinates  $(x, p) = (x^1, \dots, x^n, p_1, \dots, p_n)$  for points  $p_i dx^i$  in  $T^*U$ . In this coordinate chart we may write the ODE governing the integral curve  $\gamma(t) = (x(t), p(t))$  as

$$\begin{aligned}\dot{x}^i &= 2g^{ij}p_j \\ \dot{p}^k &= -p_i p_j \frac{\partial g^{ij}}{\partial x^k},\end{aligned}$$

known as the *Hamilton-Jacobi* equations.

- i) If  $g$  is Riemannian, consider the unit sphere bundle of the cotangent bundle

$$S(T^*M) = \{(x, p) \in T^*M : g(p, p) = 1\}.$$

Show that  $X_H$  is tangent to  $S(T^*M)$ , and prove from this that any compact Riemannian manifold is geodesically complete (i.e. every geodesic may be extended to a geodesic with domain all of  $\mathbb{R}$ ).

- ii) By substitution of the Hamilton-Jacobi equations, obtain a second order ODE of the form

$$\ddot{x}^k + \Gamma_{ij}^k \dot{x}^i \dot{x}^j = 0. \quad (3)$$

Give  $\Gamma_{ij}^k$  in terms of  $g_{ij}$  and prove that they coincide with the Christoffel symbols from the previous exercise.

- iii) Prove that Equation (3) is equivalent to the equation, for the curve  $\gamma : I \rightarrow M$ , that

$$(\gamma^* \nabla)(\gamma_* \frac{\partial}{\partial t}) = 0. \quad (4)$$

Note:  $\gamma_* \frac{\partial}{\partial t}$  is usually denoted  $\dot{\gamma}$  and is the derivative of the curve, viewed as a section of the pullback of  $TM$  via  $\gamma$ ;  $\gamma^* \nabla$  is the pullback Levi-Civita connection on  $\gamma^* TM$ . Equation (4) is called the *geodesic equation* and is the statement that the curve has parallel velocity, i.e. *it is the path of an observer experiencing no acceleration*.

- iv) Let  $f$  be a smooth function on the interval  $[a, b] \subset \mathbb{R}$  and suppose

$$\int_a^b f(x)h(x)dx = 0$$

for all smooth functions  $h$  such that  $h(a) = h(b) = 0$ . Show that  $f$  must be identically zero.

- v) Show that  $\gamma : [a, b] \rightarrow M$  is a geodesic (in any of the equivalent senses above) if and only if it is an extremum of the energy functional, which is a real-valued function on the space of paths  $\gamma : [a, b] \rightarrow M$  with fixed endpoints  $\gamma(a), \gamma(b) \in M$ , given by

$$\mathcal{E}(\gamma) = \int_a^b g(\dot{\gamma}, \dot{\gamma}) dt.$$

For simplicity you may assume that the image of  $\gamma$  lies entirely in one coordinate chart  $U \subset M$  and that the definition of *extremum* is that the derivative with respect to  $\epsilon \in \mathbb{R}$  of the usual function  $\mathbb{R} \rightarrow \mathbb{R}$

$$e(\epsilon) = \mathcal{E}(\gamma + \epsilon\sigma) \tag{5}$$

at  $\epsilon = 0$ , is zero, for all smooth curves  $\sigma : [a, b] \rightarrow \mathbb{R}^n$  with  $\sigma(a) = \sigma(b) = 0$ . Note that we are using the additive structure of  $\mathbb{R}^n$  to simplify our calculation. Explain why  $e(\epsilon)$  is in fact smooth in a neighbourhood of  $\epsilon = 0$ .

- vi) Let  $j : S^1 \times S^1 \rightarrow \mathbb{R}^3$  be the embedding of the torus given by

$$j(\theta_1, \theta_2) = \left( \left(1 + \frac{r}{R} \cos \theta_2\right) R \cos \theta_1, \left(1 + \frac{r}{R} \cos \theta_2\right) R \sin \theta_1, r \sin \theta_2 \right)$$

where  $0 < r < R$  are real constants.

Using the pullback Euclidean metric  $j^*g_{\mathbb{R}^3}$ , find a closed geodesic curve  $\gamma : S^1 \rightarrow S^1 \times S^1$  in the homotopy class of the sum of the two generating circles of  $S^1 \times S^1$ , and compute its length.

For those interested, you may take a look at Klingenberg's "Closed Geodesics on Riemannian manifolds" for a treatment of several amazing results including the existence of infinitely many prime (non-repeating) closed geodesics on any compact Riemannian manifold with finite fundamental group.